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Injective Hulls of Simple Modules Over Some Noetherian Rings

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Resumo

Estudamos a seguinte propriedade de finitude de um anel Noetheriano esquerdo R:

($\diamond$) Os envolucros injectivos de R-módulos simples são localmente Artinianos.

A propriedade ($\diamond$) é estudada em duas classes de anéis. Motivados pelo resultado de Musson: *nenhuma álgebra envolvente duma álgebra de Lie de dimensão finita solúvel mas não nilpotent sobre um corpo algebricamente fechado satisfaz a propriedade* ($\diamond$), começamos por considerar as super álgebras de Lie nilpotentes e descrevemos as de dimensão finita cuja álgebra envolvente satisfaz a propriedade ($\diamond$).

A segunda classe de anéis que estudamos são os anéis de operadores diferenciais sobre um anel de polinómios com coeficientes num corpo, $S = k[x][y;\delta]$. Para esta classe de anéis obtemos condições suficientes para a existência de extensões essenciais de módulos simples que não são Artinianos. Combinando os obtidos com resultados de Awami, Van den Bergh, e van Oystaeyen e de Alev e Dumas relativamente a classificação das extensões de Ore obtemos a caracterização completa de extensões de Ore $S = k[x][y;\sigma,\delta]$ que satisfazem a propriedade ($\diamond$).

Consideramos ainda a relaço entre teorias de torção estáveis e a propriedade ($\diamond$) em anéis Noetherianos. Em particular, mostramos que um anel Noetheriano R tem a propriedade ($\diamond$) se e só se a teoria de torção de Dickson em R-Mod é estável. Em seguida, usamos esta interpretação para obter condições suficientes para um anel Noetheriano satisfazer a propriedade ($\diamond$). Assim obtemos novos exemplos de anéis com a propriedade acima.

Abstract

We study the following finiteness property of a left Noetherian ring R:

($\diamond$) The injective hulls of simple left R-modules are locally Artinian.

We consider property ($\diamond$) for two main classes of rings. First we consider the nilpotent Lie superalgebras, motivated partly by a result of Musson on the enveloping algebras of finite dimensional solvable-but-not-nilpotent Lie algebras, which says that such an enveloping algebra does not have property ($\diamond$). We address the question of which nilpotent Lie algebras have property ($\diamond$), and give an answer in a slightly more general context of Lie superalgebras. We obtain a complete characterization of finite dimensional nilpotent Lie superalgebras over algebraically closed fields of characteristic zero, whose enveloping algebras have property ($\diamond$).

Next we study property ($\diamond$) for differential operator rings $S = k[x][y;\delta]$. We give sufficient conditions for some simple left S-modules to have non-Artinian cyclic essential extensions. This is then combined with results of Awami, Van den Bergh, and van Oystaeyen, and of Alev and Dumas on the classification of skew polynomial rings to obtain a full characterization of skew polynomial rings $S = k[x][y;\sigma,\delta]$ which have property ($\diamond$).

We also consider the stable torsion theories in connection with property ($\diamond$) for Noetherian rings. In particular, we show that a Noetherian ring R has property ($\diamond$) if and only if the Dickson torsion theory on R-Mod is stable. We then use this connection to obtain sufficient conditions for a Noetherian ring to have property ($\diamond$), and therefore obtain new examples of rings with property ($\diamond$).

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Contents

Notation

$\mathbb{N}$	The set of natural numbers.
$\mathbb{Q}$	The set of rational numbers.
$\mathbb{Z}$	The set of integers.
R	An associative ring with unit.
R-Mod	The category of left R-modules.
$\mathrm{Hom}_R(M,N)$	The set of R-homomorphisms from M to N.
$\ker f$	The kernel of a map f.
$\mathrm{End}(M)$	The ring of endomorphisms of M.
$M \subseteq_e E$	An essential extension of M.
$\mathrm{Spec}R$	The prime spectrum of the ring R.
lgl.dim(R)	The left global dimension of R.
$E(M)$	The injective hull of M.
$\mathrm{soc}(M)$	The socle of the module M.
$J(R)$	The Jacobson radical of the ring R.
$I(R)$	The set of isomorphism classes of the indecomposable injective left R-modules.
$\mathrm{Gr}S$	The associated graded ring of a filtered ring S.
$R[x;\sigma,\delta]$	The skew polynomial ring defined by R,σ, and δ.
$A_n(k)$	The nth Weyl algebra over the field k.
$\mathfrak{g}$	A fintie dimensional Lie (super)algebra.
$U(\mathfrak{g})$	The universal enveloping algebra of the Lie (super)algebra $\mathfrak{g}$.
T_τ	The class of torsion modules with respect to the torsion theory τ.

F_τ The class of torsionfree modules with respect to the torsion theory τ.

$\mathcal{D}$ The Dickson torsion theory.

$\mathcal{G}$ The Goldie torsion theory.

Introduction

Injective modules, introduced by Baer, Eckmann, and Schopf, are the building blocks in the theory of Noetherian rings. They are important tools in generalizing results on commutative algebra to noncommutative case and a starting point in this direction was the work of Matlis on injective modules over Noetherian rings [44]. Matlis showed in this paper by associating with each prime ideal P of R the injective hull $E(R/P)$ that for a commutative Noetherian ring R, indecomposable injective R-modules are in one-to-one correspondence with the prime ideals of R.

Our purpose in this work is to study a particular finiteness property on the injective hulls of simple modules over some Noetherian rings. Namely, we study Noetherian rings R such that injective hulls of simple R-modules are locally Artinian. We will denote this finiteness property by $(\diamond)$ throughout the text. This finiteness property has its roots in Jategaonkar's work on Jacobson's conjecture, and it has been studied over the years for many rings, including some stronger versions of it.

We will start with a preliminary first chapter in which we provide some background material and set some notation. We will include some facts from theory of rings and modules and of Lie algebras which will be needed in later portions of this work. Chapter 2 is devoted to remarks on the finiteness property under consideration. There we will provide the motivation for this property, and we will also list some similar properties which have appeared in the literature. This chapter also considers some examples of rings which do or do not have this finiteness property. The core material are the last three chapters. In Chapter 3 we consider property $(\diamond)$ for some Noetherian superalgebras, with a view towards injective hulls of simple modules over nilpotent Lie superalgebras. There we show that finite dimensional nilpotent Lie superalgebras $\mathfrak{g}$ over an algebraically closed field of characteristic zero whose injective hulls of simple $U(\mathfrak{g})$-modules are locally Artinian are precisely those whose even part $\mathfrak{g}_0$ is isomorphic to a nilpotent Lie algebra with an abelian ideal of codimension one, or to a direct product of an abelian Lie algebra and a certain 5-dimensional or a certain 6-dimensional nilpotent Lie algebra. In Chapter 4, we consider property $(\diamond)$ for differential operator rings. We provide sufficient conditions for a differential operator ring to have a simple module with a non-Artinian cyclic essential extension. As a consequence we characterize Ore extensions $S = K[x][y;\sigma,\delta]$ such that injective hulls of simple S-modules are locally Artinian. Chapter 5 considers property $(\diamond)$ from the view point of torsion theories. We provide a link between property $(\diamond)$ and stable torsion theories which allows us to carry the study of property $(\diamond)$ of Noetherian rings to the area of torsion theories. We use the methods of stable torsion theories, in particular the Goldie and Dickson torsion theories, to obtain sufficient conditions which guarantee $(\diamond)$ condition and to obtain new examples of Noetherian rings having property $(\diamond)$.

The contents of Chapter 3 consists of the results from a paper by the author and Christian Lomp which appeared in the *Journal of Algebra* [24], and the contents of Chapter 4 consists of results from a paper by the author, Paula A.A.B. Carvalho, and Christian Lomp [11], which has been submitted for publication.

Chapter 1

Preliminaries

This introductory chapter is a collection of definitions and results which will be referred to in the later portions of the text. For all the things that are not defined here, we refer to [47], [41], and [14].

1.1 Elementary notions

All the rings considered in this text will be associative with unit element and all modules will be unital left modules. Let R be a ring. If there exists nonzero elements $a, b \in R$ such that $ab = 0$, then a is said to be a **left zero divisor** and b is said to be a **right zero divisor**. A **zero divisor** of a ring is an element which is both a left and a right zero divisor. A ring without any left or right zero divisors is called a **domain**.

R is called a **division ring** if every nonzero element of R has a multiplicative inverse. A commutative division ring is called a **field**. The **characteristic** of a field R, denoted char(R), is the smallest positive integer p such that $p1_R = 0$. If no such integer exists we set the characteristic to be zero. Note that if the characteristic of a field is positive then it is necessarily a prime number.

By an ideal of a ring we will always mean a two sided ideal. A ring is called **simple** if it does not have any two sided ideal except the zero ideal and itself. A ring is called a **principal left (resp. right) ideal ring** if every left (resp. right) ideal of it can be generated by one element.

An element c of a ring R is called a **central element** if $cr = rc$ for all $r \in R$. The collection $Z(R)$ of all central elements of a ring is called the **center** of R.

A **prime ideal** of R is an ideal P such that for two ideals I and J of R, $IJ \subseteq P$ implies either $I \subseteq P$ or $J \subseteq P$. The collection of all prime ideals of a ring R is called the **prime spectrum** of R and is denoted by $\mathrm{Spec} R$. A **maximal (resp. left, right) ideal** of a ring is an ideal $I \neq R$ which is a maximal member of the lattice of ideals (resp. left, right ideals) of R. The **Jacobson radical** of a ring R is defined as the intersection of all left maximal ideals of R and is denoted as $J(R)$. A **simple module** is a nonzero R-module M which does not have any submodules other than itself and the zero submodule. The **socle** of a module M is defined as the sum of all simple submodules of M and we denote it by $\mathrm{soc}(M)$. R is called a **local ring** if it has a unique maximal left (right) ideal $\mathfrak{m}$. We denote a local ring with unique maximal left ideal $\mathfrak{m}$ by $(R, \mathfrak{m})$. A **semisimple** module is a module which is a direct sum of simple modules. A ring R is called **semisimple** if it is semisimple as a left module over itself. The **annihilator** of an R-module M is the set $\mathrm{Ann}_R(M) = \{r \in R \mid rM = 0\}$. A **primitive ideal** of a ring R is the annihilator of a simple R-module.

An **algebra** over a commutative ring R (or simply an R-algebra) is a ring A which is also an R-module such that $r(ab) = (ra)b = a(rb)$ for all $r \in R$ and $a, b \in A$. An ideal of an R-algebra is both an ideal of the ring A and also an R-submodule of A.

A collection $\mathcal{C}$ of subsets of a set S is said to satisfy the **ascending chain condition** if every strictly ascending chain

$$C_1 \subset C_2 \subset C_3 \subset \ldots$$

of subsets from $\mathcal{C}$ terminates after finitely many steps. A module M is called **Noetherian** if its lattice of submodules satisfies the ascending chain condition. If we use descending chains of subsets instead, we get the **descending chain condition**, and a module whose lattice of submodules satisfies the descending chain condition is called **Artinian**. Recall that for a collection $\mathcal{C}$ of sets, a set A is said to be a **maximal element** of $\mathcal{C}$ if there is no member of $\mathcal{C}$ which strictly contains A. On the other hand, a **minimal element** of such a collection is an element which does not properly contain another element of the collection. A left R-module M is Noetherian if and only if every submodule of M is finitely generated, if and only if every nonempty collection of submodules of M has a maximal element [47, (0.1.5)]. A ring R is called **left Noetherian** if it is Noetherian as a left module over itself. A **right Noetherian** ring is defined similarly. R is called **Noetherian** if it is both left and right Noetherian. It is easy to see that if M is an arbitrary module and N is a submodule of M, then M is Noetherian (resp. Artinian) if and only if N and M/N are Noetherian (resp. Artinian) [23, (1.2)]. Using this it can be showed that a finite direct sum of Noetherian modules is Noetherian [23, (1.3)]. Moreover, if R is a left Noetherian (resp. Artinian) ring, then any finitely generated left R-module is Noetherian (resp. Artinian).

In particular, when the above arguments are applied to ${}_RR$ we get the corresponding equivalent conditions on the left ideals of R which characterize left Noetherian rings. For example, $\mathbb{Z}$ is a Noetherian ring since every ideal of it is finitely generated. Also, the set of 2×2 matrices of the form $\begin{pmatrix} \mathbb{Z} & \mathbb{Q} \\ 0 & \mathbb{Q} \end{pmatrix}$ is a ring which is right Noetherian but not left Noetherian [47, (1.1.7), (1.1.9)].

A **composition series** for a module M is a chain

$$0 = M_0 < M_1 < M_2 < \ldots < M_n = M$$

of submodules of M such that the factors M_i/M_{i-1} are simple modules, for $i = 1, \ldots, n$. The number of inclusions, which in the above chain is n, is called the **length** of the composition series and the factors M_i/M_{i-1} are called the composition factors. A module which has a composition series of finite length is called a **module of finite length**.

Proposition 1.1.1 *[47, (1.3)] A module has finite length if and only if it is both Noetherian and Artinian.*

1.2 Injective modules

In this section we record the definition and some basic properties of injective modules. Main references for this section are [41] and [61].

1.2.1 Injective modules

We say that a left R-module I is **injective** if for every R-monomorphism $g : A \to B$ and every R-homomorphism $f : A \to I$ of left R-modules, there exists an R-homomorphism $h : B \to I$ such that $f = h \circ g$. In terms of diagrams, this is to say that the following diagram

$$\begin{tikzcd} & I & \\ 0 \arrow[r] & A \arrow[u, "f"] \arrow[r, "g"'] & B \arrow[ul, dashed, "\exists h"'] \end{tikzcd}$$

can be completed to a commutative triangle. We also express this property by saying that "any R-homomorphism $f : A \to I$ can be lifted to B". Alternative characterizations of injectivity is provided by the following.

Proposition 1.2.1 *[61, (2.1)] The following statements are equivalent for a left R-module E.*

(a) E is injective,

(b) given any diagram

$$\begin{tikzcd} & E & \\ 0 \arrow[r] & I \arrow[u, "f"] \arrow[r, "i"'] & R \arrow[ul, dashed, "\exists h"'] \end{tikzcd}$$

where I is a left ideal of R and $i : I \to R$ is the canonical injection, there exists an R-homomorphism $h : R \to E$ such that $f = h \circ i$, *(c) given any exact sequence*

$$0 \to A \to B \to C \to 0$$

of R-modules, the sequence

$$0 \to Hom_R(C,E) \to Hom_R(B,E) \to Hom_R(A,E) \to 0$$

is exact. In other words, $Hom_R(-,E)$ is an exact functor.

The item (b) above is known as **Baer's criterion**.

If I is an injective left R-module and M is an R-module containing I as a submodule, then the identity map $I \to I$ can be lifted to an R-homomorphism $f : M \to I$. From this it follows that we can decompose M as a direct sum $M = I \oplus \ker f$. Hence an injective module is a direct summand in every module that contains it. We record this fact along with the direct product of injective modules in the following. We note that a monomorphism $f : M \to N$ of left R-modules is said to **split** if $\mathrm{Im}(f)$ is a direct summand of N.

Proposition 1.2.2 *[41, (3.4)] (1) A direct product $I = \prod_\alpha I_\alpha$ of left R-modules is injective if and only if each I_α is injective. (2) A left R-module I is injective if and only if any monomorphism $I \to M$ of left R-modules splits in R-mod.*

It follows from the above results that a finite direct sum of injective modules is again injective. However, in general an arbitrary direct sum of injective modules is not injective. This feature actually characterizes Noetherian rings, by the following result which is due to Bass and Papp:

Theorem 1.2.3 *[41, (3.46)] For any ring R, the following statements are equivalent:*

(a) Any direct limit of injective left R-modules is injective.

(b) Any direct sum of injective left R-modules is injective.

(c) Any countable direct sum of injective left R-modules is injective.

(d) R is a left Noetherian ring.

Example 1.2.4 *A division ring has only two left ideals, zero and itself, hence any module over a division ring automatically satisfies the Baer's criterion. Therefore every left module over a division ring is injective. In particular, every vector space is injective. Rings over which every left module is injective are precisely the semisimple rings [40, (2.9)]. For example, as a module over itself, the ring of integers $\mathbb{Z}$ is not injective since the map $f : 2\mathbb{Z} \to \mathbb{Z}$ given by $f(2n) = n$ cannot be lifted to a homomorphism $f' : \mathbb{Z} \to \mathbb{Z}$.*

1.2.2 Injective hulls

An extension $M \subseteq E$ of left R-modules is said to be an **essential extension** of M if for every nonzero submodule N of E we have $M \cap N \neq 0$. We write $M \subseteq_e E$ to indicate that E is an essential extension of M and also say that M is an essential submodule of E. Observe that this is equivalent to say that for every nonzero element e of E, the intersection $Re \cap M$ is nonzero.

For example, if R is a commutative domain with field of fractions Q, then $R \subseteq_e Q$ as R-modules. Also, essential extensions satisfy the "transitivity" property so that $M \subseteq_e E$ and $E \subseteq_e E'$ imply $M \subseteq_e E'$.

An R-module E is said to be a **maximal essential extension** of an R-module M if E is an essential extension of M and M is not essential in any proper extension of E. An R-module I is said to be a **minimal injective extension** of an R-module M if I is injective and no proper submodule of I which contains M is injective. The following result is due to Eckman, Schöpf and Baer.

Proposition 1.2.5 *Let M be an R-module and E be an R-module extension of M. Then the following statements are equivalent.*

(a) E is injective and is an essential extension of M.

(b) E is a maximal essential extension of M.

(c) E is a minimal injective extension of M.

A proof of this can be found in [61, (2.20)] or [41, (3.30)]. For an R-module M, an R-module E which satisfies the equivalent conditions of Proposition 1.2.5 is called an **injective hull** (or **injective envelope**) of M. We denote an injective hull of an R-module M by $E(M)$.

The immediate question whether every module has an injective hull has an affirmative answer. We show this by giving an outline of the construction of an injective hull of a given module M.

We start with divisible modules. A left R-module D is said to be **divisible** if $rD = D$ for every element r of R which is not a zero divisor. Every injective module is divisible [61, (2.6)]. The converse is true if R is a principal ideal domain [61, (2.8)]. In particular, viewed as modules over the integers, an abelian group is divisible if and only if it is injective.

Example 1.2.6 *$\mathbb{Q}$ is injective as a $\mathbb{Z}$-module since it is divisible. Moreover $\mathbb{Z} \subseteq_e \mathbb{Q}$ and it follows that $E(\mathbb{Z}) = \mathbb{Q}$ as $\mathbb{Z}$-modules. More generally, if R is a commutative domain with quotient field Q then $E(R) = Q$ as R-modules.*

We begin by recording that every module can be embedded in an injective module. First we recall the following result from abelian group theory:

Lemma 1.2.7 *[61, (2.12)] Every abelian group can be embedded in an injective abelian group.*

Hence M can be embedded as an abelian group in an injective abelian group, say I. Then, since the $\mathrm{Hom}_{\mathbb{Z}}(R, -)$ is left exact, it follows that there is an embedding (as abelian groups) $\mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{Z}(R, I)$.

For an arbitrary abelian group G, the abelian group $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ can be given the structure of a left R-module as follows: for $r \in R$ and $f \in \mathrm{Hom}_{\mathbb{Z}}(R, G)$, let rf be defined by $(rf)(s) = f(sr)$. Moreover, with this R-module structure, the above embedding of hom sets is actually an R-homomorphism.

We can embed M in the R-module $\mathrm{Hom}_{\mathbb{Z}}(R, M)$ by defining $\phi : M \to \mathrm{Hom}_{\mathbb{Z}}(R, M)$ by

$\phi(m)(r) = rm$ [61, (2.14)]. Hence we arrive at a sequence of embeddings of R-modules

$$M \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, M) \hookrightarrow \mathrm{Hom}_{\mathbb{Z}}(R, I).$$

We make the final comment by noting that if G is an injective abelian group then with the above R-module structure $\mathrm{Hom}_{\mathbb{Z}}(R, G)$ is injective (for a proof see [61, (2.13)]). Hence $\mathrm{Hom}_{\mathbb{Z}}(R, I)$ is the desired injective R-module for M. We note this as a separate theorem. A detailed proof of this construction can be found in [61, §2.3]:

Lemma 1.2.8 *Every module can be embedded in an injective module.*

Now that we know every R-module can be embedded in an injective R-module, we apply Zorn's Lemma to reach our desired conclusion.

Lemma 1.2.9 *Every R-module has a maximal essential extension. In other words, every R-module has an injective hull.*

Proof: Embed M in an injective module I. Consider the collection $\{E \in R\text{-Mod} \mid M \subseteq_e E \leq I\}$ of submodules of I. This is nonempty since it contains M. For any chain of modules from this collection, the union of the chain is also an essential extension of M. By Zorn's Lemma, there exists a submodule E of I which is maximal with respect to the property that $M \subseteq_e E \subseteq I$. We claim that E is a maximal essential extension of M. If it is not, we have an embedding $E \subsetneq E'$ such that $M \subseteq_e E'$. Since I is injective, the embedding $E \hookrightarrow I$ can be lifted to a homomorphism $g : E' \to I$. Since $\ker g \cap M = 0$, the essentiality of M in E' implies that $\ker g = 0$. Hence E' can be identified with its image $g(E)$. But this means $M \subseteq_e E'$, contradicting the maximality of E. □

The following is a collection of some basic properties of injective hulls.

Lemma 1.2.10 *[41, (3.32) & (3.33)] (1) Any two injective hulls E and E' of a module M are isomorphic. (2) If I is an injective module containing M, then I contains a copy of $E(M)$. (3) Any essential extension $M \subseteq_e N$ can be enlarged into a copy of $E(M)$. Indeed, if $M \subseteq_e N$ then $E(M) = E(N)$.*

Since two injective hulls of a module M are isomorphic by the above result, we can refer to $E(M)$ as *the* injective hull of M. Another result concerning injective modules over Noetherian rings is the following. We call a module M **indecomposable** if it is not a direct sum of two nonzero submodules.

Theorem 1.2.11 *For any ring R, the following are equivalent:*

(a) R is left Noetherian.

(b) Any injective left R-module is a direct sum of indecomposable (injective) submodules.

(c) There exists a cardinal number α such that any injective left R-module M is a direct sum of (injective) submodules of cardinality $\leq \alpha$.

For a proof we refer to [41, (3.48)] where the equivalence $(1) \Leftrightarrow (2)$ is attributed to Matlis and Papp while the equivalence $(1) \Leftrightarrow (3)$ is attributed to Faith. When working with finitely generated modules over Noetherian rings the following result is often handy. Its proof follows easily from $(1) \Rightarrow (2)$ of Theorem 1.2.11.

Corollary 1.2.12 *Let N be a finitely generated left module over a left Noetherian ring R. Then $E(N)$ is a finite direct sum of indecomposable injective modules.*

An important result in the study of injective modules over Noetherian rings is the work of Matlis , who gave a complete list of indecomposable injective modules over commutative Noetherian rings. For a ring R, we denote by $I(R)$ the set of isomorphism classes of indecomposable injective left R-modules.

Theorem 1.2.13 *[44, (3.1)] Let R be a commutative Noetherian ring. There is a one to one correspondence between the prime ideals of R and the set $I(R)$ given by $P \mapsto E(R/P)$ for every prime ideal P of R.*

We close this section by a characterization of indecomposable injective modules. A nonzero left R-module U is called **uniform** if any two nonzero submodules of U have a nonzero intersection. This is equivalent to say that nonzero submodules of U are indecomposable, or that any nonzero submodule of U is essential in U. A left ideal I of a ring R is called **meet irreducible** if R/I is a uniform left R-module.

Then, for an injective module, being indecomposable, being uniform, and being the injective hull of a cyclic uniform module are all the same [41, (3.52)].

1.3 Filtered and graded algebraic structures

Some of the rings we will deal with are filtered and/or graded rings and in this section we gather some facts about them. Our main reference for this section is [47, (1.6)].

Let S be a ring. A family $\{F_i\}_{i\in\mathbb{N}}$ of additive subgroups of S is said to be a **filtration** of S if it satisfies the following properties:

(i) for each i, j we have $F_iF_j \subseteq F_{i+j}$,

(ii) for $i < j$, $F_i \subseteq F_j$ and

(iii) $\cup F_i = S$.

When S has such a filtration, it is called a **filtered ring**. An $\mathbb{N}$**-graded ring** is a ring T with a family $\{T_i\}_{i\in\mathbb{N}}$ of additive subgroups of T satisfying

(i) $T_iT_j \subseteq T_{i+j}$, and

(ii) $T = \bigoplus_{i=0}^{\infty} T_i$, as an abelian group.

When this is the case, the family $\{T_i\}$ is called an $\mathbb{N}$**-grading** of the ring T. A nonzero element of T which belongs to T_n for some n is called a **homogeneous** element of degree n.

Any graded ring T has a natural filtration $\{F_n\}$ with $F_n = T_0 \oplus \ldots \oplus T_n$. If S is a filtered ring, then we can construct a graded ring from S in the following way. For any n, we set $T_n = F_n/F_{n-1}$ and $T = \bigoplus T_n$. We define a multiplication on T as follows. For any homogeneous element $a \in F_n \backslash F_{n-1}$, we define the degree of a to be n and the element $\overline{a} = a + F_{n-1}$ is called the leading term of a. Let c be another homogeneous element of degree m. Then the multiplication $\overline{ac}$ is defined to be $ac+F_{m+n-1}$. This is well-defined and makes T into a graded ring which we denote by $\mathrm{Gr}S$ and call the **associated graded ring**.

Associated graded rings are useful in the sense that some properties can be transferred from $\mathrm{gr}S$ to S. For example:

Proposition 1.3.1 *Let S be a filtered ring. Then*

(a) If GrS is an integral domain then S is an integral domain-

(b) If GrS is prime then S is prime.

(c) If GrS is left Noetherian then S is left Noetherian.

A proof of the above properties can be found in [47, (1.6.6) and (1.6.9)].

Let k be a field and G be a group. A G**-graded vector space** V is a k-vector space together with a family $\{V_g\}_{g\in G}$ of subspaces such that $V = \bigoplus_G V_g$. An element v of a G-graded vector space V is said to be **homogeneous** of degree g if $v \in V_g$ for some $g \in G$. We denote the degree of a homogeneous element v by $|v|$. Every element v of a G-graded vector space V has a unique decomposition of the form $v = \sum_{g\in G} v_g$ where the element v_g is called the **homogeneous component** of v of degree g.

A subspace U of a G-graded vector space V is said to be G**-graded** if it contains the homogeneous components of each of its elements, in other words, if $U = \bigoplus_G (U \cap V_g)$. For two G-graded vector spaces V and W, a linear map $\alpha : V \to W$ is called **homogeneous** of degree g, $g \in G$, if $\alpha(V_h) \subset W_{g+h}$ for all $h \in G$. The mapping $g : V \to W$ is called a homomorphism of the G-graded vector spaces if g is homogeneous of degree 0.

A G**-graded algebra** is a k-algebra A whose underlying k-vector space is G-graded, *i.e.* $A = \bigoplus_G A_g$ and such that $A_g A_h \subseteq A_{g+h}$ for all $g, h \in G$. When there is no danger of confusion, we will shortly say graded algebra instead of G-graded algebra. It follows from the definition that if A is a graded algebra then A_0 is a subalgebra of A. A homomorphism of G-graded algebras is an algebra homomorphism as well as a homomorphism of G-graded vector spaces. This means in particular that a homomorphism is homogeneous of degree 0.

For any additive subgroup X of a graded algebra A, we set $X_g = X \cap A_g$ and we say that X is graded when $X = \bigoplus_G X_g$. We say that a left ideal I of a graded algebra A is a **graded left ideal** if the underlying additive group of I is graded. That is, an ideal I is graded if it contains the homogeneous components of each of its elements. One defines similarly the notions of graded right or two sided ideals and of a graded subring. Note that whenever I is a graded ideal of a graded algebra A, then A/I is also graded with the grading given by $(A/I)_g = (A_g + I)/I$.

For a graded algebra A, a left A-module M is said to be a **graded module** if M is graded as a vector space and moreover $A_g M_h \subset M_{g+h}$ for all $g, h \in G$. A homomorphism of graded A-modules is both a homomorphism of A-modules and also of graded vector spaces. This means that it must be homogeneous of degree 0 and it is A-linear.

1.4 Some Noetherian rings

In this section we define and give some basic properties of some algebraic structures and classes of Noetherian rings.

1.4.1 Lie algebras

In this section we introduce the necessary ingredients from the theory of Lie algebras. For more information we refer to [14] and [29].

Let k be a field. A **Lie algebra** over k is a k-vector space $\mathfrak{g}$ together with a bilinear map $[\ ,\ ] : \mathfrak{g} \times \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, which satisfies the following properties:

(i) $[x, x] = 0$ for all $x \in \mathfrak{g}$.

(ii) $[x, [y, z]] + [y, [z, x]] + [z, [x, y]] = 0$ for all $x, y, z \in \mathfrak{g}$.

The identity in (ii) of the above definition is called the **Jacobi identity**. Observe that bilinearity of the bracket and (i) above imply together the anticommutativity property: for all $x, y \in \mathfrak{g}$ we have $[x, y] = -[y, x]$. Moreover, if the characteristic of the field k is not equal to 2, anticommutativity also implies (i).

It is easy to see from the definition that in a Lie algebra $\mathfrak{g}$ we have $[0, x] = 0$ for all $x \in \mathfrak{g}$ and if $x, y \in \mathfrak{g}$ satisfy $[x, y] \neq 0$ then it follows that x and y are linearly independent.

A Lie algebra $\mathfrak{g}$ is called **abelian** if $[x, y] = 0$ for all $x, y \in \mathfrak{g}$. The **dimension** of the Lie algebra $\mathfrak{g}$ is the vector space dimension of $\mathfrak{g}$. A Lie algebra **homomorphism** from $\mathfrak{g}$ to $\mathfrak{h}$ is a vector space homomorphism $f : \mathfrak{g} \to \mathfrak{h}$ which satisfies $f([x, y]) = [f(x), f(y)]$ for all $x, y \in \mathfrak{g}$. Lie algebras $\mathfrak{g}$ and $\mathfrak{h}$ are called isomorphic if there exists a Lie algebra homomorphism which is an isomorphism of vector spaces. A **Lie subalgebra** of a Lie algebra $\mathfrak{g}$ is a subspace $\mathfrak{h}$ of $\mathfrak{g}$ such that $[x, y] \in \mathfrak{h}$ for all $x, y \in \mathfrak{h}$. We define an **ideal** to be a subspace I of $\mathfrak{g}$ for which $[x, y] \in I$ for all $x \in \mathfrak{g}$ and $y \in I$. If I is an ideal of $\mathfrak{g}$, then the quotient space $\mathfrak{g}/I$ becomes a Lie algebra by defining the bracket of two elements as $[x + I, y + I] = [x, y] + I$ for all $x, y \in \mathfrak{g}$.

Any algebra A becomes a Lie algebra if we define the bracket of two elements x, y of A to be $[x, y] = xy - yx$. The element $xy - yx$ is called the **commutator** of x and y. For example, if V is a finite dimensional vector space, then the Lie algebra structure defined on $\mathrm{End}(V)$ is called the **general linear algebra** and we denote it by $\mathfrak{gl}(V)$ or $\mathfrak{gl}(n, k)$ to distinguish it from the ring $\mathrm{End}(V)$. Note that the sets of all upper triangular matrices, strictly upper triangular matrices, and all diagonal matrices are all subalgebras of $\mathfrak{gl}(n)$.

A **derivation** of a not necessarily associative k-algebra A is a k-linear map $d : A \to A$ which satisfies $d(xy) = xd(y) + d(x)y$ for all $x, y \in A$. The set $D(A)$ of all derivations of A is a subspace of $\mathrm{End}(A)$. Moreover, the commutator of two derivations is also a derivation, hence $D(A)$ is a Lie subalgebra of $\mathfrak{gl}(A)$.

In particular, one can define the notion of the derivation of a Lie algebra, since a Lie algebra is an algebra in the above sense. If $\mathfrak{g}$ is a Lie algebra, for any element x of $\mathfrak{g}$ we define the **adjoint action** of x on $\mathfrak{g}$ by $\mathrm{ad}_x(y) = [x, y]$ for all $y \in \mathfrak{g}$. With the help of the Jacobi identity, one can show that for every element x of $\mathfrak{g}$, the map ad_x is a derivation of $\mathfrak{g}$. Derivations of $\mathfrak{g}$ arising in this way are called the **inner** derivations while others are called **outer**. The map $\mathfrak{g} \to D(\mathfrak{g})$ given by $x \mapsto \mathrm{ad}_x$ is called the **adjoint representation** of $\mathfrak{g}$.

1.4.1.1 Solvable and nilpotent Lie algebras

Let $\mathfrak{g}$ be a Lie algebra. The **derived algebra** of $\mathfrak{g}$ is the algebra $[\mathfrak{g},\mathfrak{g}]$ which is the span of all elements of the form $[x,y]$ with $x,y\in\mathfrak{g}$. We define a sequence of ideals of $\mathfrak{g}$ in the following way. Let $\mathfrak{g}^{(0)}=\mathfrak{g}$ and $\mathfrak{g}^{(1)}=[\mathfrak{g},\mathfrak{g}]$, and inductively we define $\mathfrak{g}^{(i)}=[\mathfrak{g}^{(i-1)},\mathfrak{g}^{(i-1)}]$. This is called the **derived series** of $\mathfrak{g}$. We say that $\mathfrak{g}$ is **solvable** if $\mathfrak{g}^{(n)}=0$ for some n.

We define another sequence of ideals by first letting $\mathfrak{g}^0=\mathfrak{g}$ and $\mathfrak{g}^1=[\mathfrak{g},\mathfrak{g}]$, $\mathfrak{g}^2=[\mathfrak{g},\mathfrak{g}^1]$ and $\mathfrak{g}^n=[\mathfrak{g},\mathfrak{g}^{n-1}]$. The sequence of ideals we just defined is called the **descending central series** or **lower central series**. $\mathfrak{g}$ is called **nilpotent** if $\mathfrak{g}^n=0$ for some n. If $\mathfrak{g}$ is a nilpotent Lie algebra, the least positive integer r such that $\mathfrak{g}^r=0$ is called the **nilpotency degree** of $\mathfrak{g}$.

Obviously, for every i we have $\mathfrak{g}^{(i)}\subset\mathfrak{g}^i$ and so nilpotent Lie algebras are solvable. Moreover, a Lie algebra $\mathfrak{g}$ is solvable if and only if there exists a chain of subalgebras

$$\mathfrak{g}=\mathfrak{g}_0\supset\mathfrak{g}_1\supset\mathfrak{g}_2\supset\ldots\supset\mathfrak{g}_k=0$$

such that $\mathfrak{g}_{i+1}$ is an ideal of $\mathfrak{g}_i$ and such that each quotient $\mathfrak{g}_i/\mathfrak{g}_{i+1}$ is abelian.

1.4.1.2 Universal enveloping algebra of a Lie algebra

Let V be a finite dimensional vector space over a field k. Let $T^0(V)=k$, $T^1(V)=V$, $T^2(V)=V\otimes V$ and generally $T^m(V)=V\otimes\ldots\otimes V$ (m copies). Let $T(V)=\oplus_{i=0}^{\infty}T^i(V)$. Then the multiplication defined on the homogeneous generators on $T(V)$ by

$$(v_1\otimes\ldots\otimes v_n)(w_1\otimes\ldots\otimes w_m)=v_1\otimes\ldots\otimes v_n\otimes w_1\otimes\ldots\otimes w_m\in T^{m+n}(V)$$

makes $T(V)$ an associative algebra. This algebra is called the **tensor algebra** on V.

Now let I be the two sided ideal in $T(V)$ generated by all elements $x\otimes y-y\otimes x$, $x,y\in V$. The factor algebra $S(V)=T(V)/I$ is called the **symmetric algebra** on V.

The definition of the universal enveloping algebra of a Lie algebra is as follows. A **universal enveloping algebra** of a Lie algebra $\mathfrak{g}$ is a pair (U, i) where U is an associative algebra with unit over k, and $i : \mathfrak{g} \to U$ is a linear map satisfying

$$i([x,y]) = i(x)i(y) - i(y)i(x) \tag{1.1}$$

for all $x, y \in \mathfrak{g}$ and with the following universal property: for any associative k algebra A with unit and any linear map $j : \mathfrak{g} \to A$ satisfying Equation 1.1, there exists a unique algebra homomorphism $\phi : U \to A$ such that $\phi \circ i = j$. It follows from this universal property that such a pair is unique if it exists.

The existence of a universal enveloping algebra is guaranteed by the following construction. Let $\mathfrak{g}$ be a Lie algebra. Let $T(\mathfrak{g})$ be the tensor algebra on $\mathfrak{g}$ and let I be the ideal generated by the elements $x \otimes y - y \otimes x - [x,y]$, $x, y \in \mathfrak{g}$. Define $U(\mathfrak{g}) = T(\mathfrak{g})/I$. Let $\pi : T(\mathfrak{g}) \to U(\mathfrak{g})$ be the canonical homomorphism. Then if $i : \mathfrak{g} \to U(\mathfrak{g})$ is the restriction of π to $\mathfrak{g}$, then the pair $(i, U(\mathfrak{g}))$ is a universal enveloping algebra of $\mathfrak{g}$.

The following result is known as the **Poincaré-Birkhoff-Witt Theorem**. Its proof can be found in [29, (17.3)].

Theorem 1.4.1 *Let $\mathfrak{g}$ be a Lie algebra with an ordered basis $\{x_1, x_2, \ldots\}$. Then the elements $x_{i(1)} \ldots x_{i(m)} = \pi(x_{i(1)} \otimes \ldots \otimes x_{i(m)})$, $m \in \mathbb{Z}^+$, $i(1) \leq i(2) \leq \ldots \leq i(m)$, along with 1, form a basis for $U(\mathfrak{g})$.*

We will shortly refer to a basis of $U(\mathfrak{g})$ constructed in the above sense as a **PBW-basis**.

Let $\mathfrak{g}$ be a Lie algebra and let $U(\mathfrak{g})$ be its enveloping algebra. For any integer n, let $U_n(\mathfrak{g})$ denote the vector subspace of $U(\mathfrak{g})$ generated by the products $x_1 x_2 \ldots x_p$ where $x_1, x_2, \ldots, x_p \in \mathfrak{g}$ and $p \leq n$. Then $\{U_n(\mathfrak{g})\}$ is an increasing sequence whose union is $U(\mathfrak{g})$ and it satisfies

$$U_0(\mathfrak{g}) = k, \quad U_1(\mathfrak{g}) = k \cdot 1 \oplus \mathfrak{g}, \quad U_n(\mathfrak{g})U_m(\mathfrak{g}) \subset U_{m+n}(\mathfrak{g}).$$

Hence the sequence $\{U_n(\mathfrak{g})\}$ is a filtration of $U(\mathfrak{g})$ which is called the **canonical filtration**.

If $\{x_i \mid i \in I\}$ is a basis for the Lie algebra $\mathfrak{g}$, then the associated graded ring $\mathrm{Gr}U(\mathfrak{g})$ obtained from the canonical filtration is a commutative k-algebra generated by $\{\overline{x_i} \mid i \in I\}$. Since any finitely generated commutative k-algebra is Noetherian, it follows that the associated graded ring of $U(\mathfrak{g})$ is Noetherian whenever $\mathfrak{g}$ is finite dimensional. Hence the universal enveloping algebra of a finite dimensional Lie algebra is Noetherian by Proposition 1.3.1(3) (see [47, 1.7.4]).

1.4.2 Lie superalgebras

We continue with the fundamental notions of Lie superalgebras. Our main references for this subsection is [60] and [4].

A $\mathbb{Z}/2\mathbb{Z}$-graded algebra of the form $A = A_0 \oplus A_1$ is called a **superalgebra**. We apply the general definitions given for general graded algebraic structures for the case of a superalgebra without any change, except in a superalgebra A, the elements of the subalgebra A_0 will be called **even** while the elements of the subspace A_1 will be called **odd**. If A is a superalgebra, then the map $a = a_0 + a_1 \mapsto a_0 - a_1$ is an involution of A. Likewise, we call a $\mathbb{Z}/2\mathbb{Z}$-graded vector space a **super vector space**.

A **Lie superalgebra** over a field k is a super vector space $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ provided with a multiplication $[\ ,\ ] : \mathfrak{g} \otimes \mathfrak{g} \to \mathfrak{g}$, called the **Lie bracket**, such that

(i) The bracket is superantisymmetric (or graded skew symmetric), *i.e.*, $[x,y] = -(-1)^{|x||y|}[y,x]$ for all nonzero homogeneous elements $x,y \in \mathfrak{g}$.

(ii) The bracket satisfies the **super Jacobi identity**, *i.e.*,

$$(-1)^{|x||z|}[x,[y,z]] + (-1)^{|y||x|}[y,[z,x]] + (-1)^{|z||y|}[z,[x,y]] = 0.$$

The adjoint action of an element x of a Lie superalgebra $\mathfrak{g}$ is defined similar to the case of a Lie algebra, by defining $\mathrm{ad}_x : \mathfrak{g} \to \mathfrak{g}$ as $\mathrm{ad}_x(a) = [x,a]$ for all $a \in \mathfrak{g}$. It follows from the definition that in a Lie superalgebra $\mathfrak{g}$, the subalgebra $\mathfrak{g}_0$ is itself a Lie algebra and the odd part $\mathfrak{g}_1$ is a module over $\mathfrak{g}_0$. The definitions of a graded subalgebra, a graded ideal, a graded quotient algebra of $\mathfrak{g}$ are easily adopted from the respective definitions given above for general graded algebras.

1.4.2.1 Solvable and nilpotent Lie superalgebras

As in the "nonsuper" case, we define the solvable and nilpotent Lie superalgebras by means of the vanishing of the lower central and derived series. The **lower central series** is defined to be the sequence of ideals $\mathfrak{g}$ defined by $\mathfrak{g}^0 = \mathfrak{g}$, $\mathfrak{g}^1 = [\mathfrak{g}, \mathfrak{g}]$ and generally $\mathfrak{g}^i = [\mathfrak{g}, \mathfrak{g}^{i-1}]$. Also the **derived series** of $\mathfrak{g}$ is defined as $\mathfrak{g}^{(0)} = \mathfrak{g}$, $\mathfrak{g}^{(1)} = [\mathfrak{g}, \mathfrak{g}]$ and generally $\mathfrak{g}^{(i)} = [\mathfrak{g}^{(i-1)}, \mathfrak{g}^{(i-1)}]$. $\mathfrak{g}$ is called **solvable** if its derived series vanishes for some n. It is called **nilpotent** if its lower central series vanishes for some n. As in the nonsuper case, nilpotent Lie superalgebras are solvable.

The solvability of a Lie superalgebra is determined by its even part:

Proposition 1.4.2 *[60, Proposition 2(a), p. 236] A Lie superalgebra $\mathfrak{g}$ is solvable if and only if its Lie algebra $\mathfrak{g}_0$ is solvable.*

A similar result for nilpotency is also available.

Proposition 1.4.3 *[26, Corollary 2] A Lie superalgebra $\mathfrak{g}$ is nilpotent if and only if ad_x is a nilpotent operator for every homogeneous element $x \in \mathfrak{g}$. Consequently, a Lie superalgebra $\mathfrak{g}$ is nilpotent if and only if $\mathfrak{g}_0$ is a nilpotent Lie algebra and the action of $\mathfrak{g}_0$ on $\mathfrak{g}_1$ is by nilpotent operators.*

1.4.2.2 Enveloping algebra of a Lie superalgebra

The definition of a universal enveloping algebra of a Lie superalgebra is similar to the standard case. Let $\mathfrak{g}$ be a Lie superalgebra over a field k. An associative k-algebra U with unit and with a linear map $\sigma : \mathfrak{g} \to U$ is called a **universal enveloping algebra** of $\mathfrak{g}$ if

(i) for any $x \in \mathfrak{g}_i$, $y \in \mathfrak{g}_j$, with $i, j \in \{0, 1\}$,

$$\sigma([x, y]) = \sigma(x)\sigma(y) - (-1)^{|x||y|}\sigma(y)\sigma(x)$$

and

(ii) for any associative k-algebra A with unit and a k-linear map $\sigma' : \mathfrak{g} \to A$ which satisfies (i), there is a unique algebra homomorphism $r : U \to A$ such that $r(1) = 1$ and $r \circ \sigma = \sigma'$.

We briefly give the construction of an enveloping algebra of a Lie superalgebra. For the details we refer to [60, §2]. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and let $T(\mathfrak{g})$ be the tensor algebra of the vector space $\mathfrak{g}$. We let J be the ideal of the tensor algebra generated by the elements of the form

$$x \otimes y - (-1)^{|x||y|} y \otimes x - [x, y]$$

for $x, y \in \mathfrak{g}$. These elements are homogeneous of degree $|x| + |y|$ and so J is a graded ideal. We let

$$U(\mathfrak{g}) = T(\mathfrak{g})/J.$$

$U(\mathfrak{g})$ is an associative superalgebra which satisfies the necessary conditions of the above definition and hence it is the universal enveloping algebra of $\mathfrak{g}$. In particular, the natural mapping $\mathfrak{g} \to U(\mathfrak{g})$ is injective and we can identify $\mathfrak{g}$ with a graded subspace of $U(\mathfrak{g})$.

Let $\mathfrak{g}$ and $\mathfrak{g}'$ be two Lie superalgebras and let σ (resp. σ') be the natural mapping of $\mathfrak{g}$ into its enveloping algebra $U(\mathfrak{g})$ (resp. $U(\mathfrak{g}')$). If $f : \mathfrak{g} \to \mathfrak{g}'$ is a Lie algebra homomorphism, then the universal property of the enveloping algebra of a Lie superalgebra implies that there exists a unique homomorphism $\overline{f} : U(\mathfrak{g}) \to U(\mathfrak{g}')$ of superalgebras such that $\sigma' \circ f = \overline{f} \circ \sigma$, $\overline{f}(1) = 1$ [60, Corollary 2, p. 20].

Let $\mathfrak{g}$ be a finite dimensional Lie superalgebra. We identify the elements of $\mathfrak{g}$ with their images in its enveloping algebra $U = U(\mathfrak{g})$. The enveloping algebra $U = U(\mathfrak{g})$ has the following filtration: $U^0 = k$, $U^1 = k + \mathfrak{g}$, and generally we define U^n to be the subspace of U generated by all monomials of degree less than or equal to n. This is actually a filtration of U and the associated graded algebra of U is the tensor product

$$\mathrm{Gr} U = k[x_1, \ldots, x_m] \otimes_k \wedge(y_1, \ldots, y_n)$$

where m and n are the vector space dimensions of the even and odd parts of $\mathfrak{g}$, respectively and $\wedge(y_1,\ldots,y_n)$ is the exterior algebra on n letters which is defined as the quotient $T(V)/I$ where V is an n-dimensional vector space and I is the ideal of the tensor algebra generated by all elements of the form $x\otimes x$ where $x\in V$. Hence the enveloping algebra of a finite dimensional Lie superalgebra is Noetherian (see [60, §2.3, p. 25].

Finally, we have the super version of the Poincaré-Birkhoff-Witt theorem:

Theorem 1.4.4 *[60, Theorem 1, p.26] Let $\mathfrak{g}$ be a Lie superalgebra with an ordered basis $\{x_1,\ldots,x_n\}$ consisting of homogeneous elements. Then the set of all products of the form $x_1^{p_1}\cdots x_n^{p_n}$, where $x_i^0=1, p_i\geq 0$ and $p_i\leq 1$ whenever x_i is odd, is a basis of $U(\mathfrak{g})$.*

1.4.3 Skew polynomial rings

Let R be a ring and let σ be an automorphism of R. An additive endomorphism δ of R is said to be a **σ-derivation** of R if it satisfies

$$\delta(ab)=\delta(a)\sigma(b)+a\delta(b)$$

for all $a,b\in R$. If σ is the identity map, a σ-derivation is simply referred to as a derivation. Note the definition of a derivation we give here for a ring is different than the one we gave for algebras over fields. While we required a derivation for a k-algebra to be k-linear, we only require a derivation of a ring to be $\mathbb{Z}$-linear.

For a ring R, with an automorphism σ and a σ-derivation δ of R, we define the **skew polynomial ring** attached to this data to be the free left R-module with basis $1,x,x^2,\ldots$ whose multiplication is defined by the rules $xr=\sigma(r)x+\delta(r)$ and $x^ix^j=x^{i+j}$. We denote this ring by $R[x;\sigma,\delta]$. If $\delta=0$ we write $R[x;\sigma]$ and if σ is the identity map we write $R[x;\delta]$.

Some properties of the ring R are reflected in the skew polynomial ring $S=R[x;\sigma,\delta]$. For example, if R is a domain, a prime ring, or left (or right) Noetherian then so is S. The proof of these facts and also the detailed skew polynomial ring construction can be found in [47, §1.2].

1.4.4 Weyl algebras

Let k be a field. Let $A_n(k)$ be the k-algebra generated by $x_1, x_2, \ldots, x_n, y_1, y_2, \ldots, y_n$ subject to the relations

$$x_i y_j - y_j x_i = \delta_{ij}$$

and

$$x_i x_j - x_j x_i = y_i y_j - y_j y_i = 0.$$

We call $A_n(k)$ the n**th Weyl algebra** over k.

Alternatively, $A_n(k)$ can be realized as an iterated skew polynomial ring in the following way. Let $R = k[x_1, x_2, \ldots, x_n]$ be the commutative polynomial ring. We define the rings

$$R_0 = R,\ R_{i+1} = R_i[y_{i+1}; \partial/\partial x_i].$$

Then the k-algebra R_n has the generators which satisfy the relations of the Weyl algebra and conversely, the generators of the Weyl algebra $A_n(k)$ satisfy the relations for R_i [47, §1.3]. Moreover, if the characteristic of the field k is zero, then $A_n(k)$ is a simple Noetherian integral domain [47, Theorem 1.3.5]. This is not true anymore in positive characteristic, since in that case if the characteristic of the field is m, then the element x_i^m is central and generates a nonzero ideal.

1.5 Krull and global dimension of rings

In this section we define two ring theoretical dimensions which will appear in our work.

1.5.1 Krull dimension

For a commutative ring R its Krull dimension is defined to be the maximum possible length of a chain $P_0 \subset P_1 \subset \ldots \subset P_n$ of distinct prime ideals of R. We say that the ring R has infinite Krull dimension if it has arbitrarily long chains of distinct prime ideals.

In the noncommutative case, the Krull dimension is defined in terms of deviation of a poset. Let A be a poset and for $a, b \in A$ let $a/b = \{x \in A \mid a \geq x \geq b\}$. This is a subposet of A and is called the factor of a by b. We say that the poset A satisfies the descending chain condition if every descending chain in A becomes stationary.

The deviation $\mathrm{dev}A$ of a poset A is defined as follows. If A is trivial (*i.e.* a partially ordered set which has no two distinct, comparable elements), we let $\mathrm{dev}A = -\infty$ and if A is a nontrivial poset which satisfies the descending chain condition then we let $\mathrm{dev}A = 0$. For an ordinal α we define the deviation of A to be α if

(i) $\mathrm{dev}A \neq \beta < \alpha$,

(ii) in any descending chain $a_1 > a_2 > a_3 > \ldots$ of elements of A all but finitely many factors of a_i by a_{i+1} have deviation less than α.

Let M be a module and let $\mathcal{L}(M)$ be the lattice of submodules of M. The **Krull dimension** of M, denoted by $\mathrm{K.dim}M$, is the deviation of the poset $\mathcal{L}(M)$ when it exists. The left Krull dimension of a ring R is the Krull dimension of R as a left module over itself, denoted $\mathrm{lK.dim}R$. In particular, M is Artinian if and only if its Krull dimension is zero.

If M is a Noetherian module then its Krull dimension exists. Also if R is a left Noetherian ring, its left Krull dimension exists [47, 6.2.3].

We list three results which we will need later in the text when dealing with Krull dimension.

Lemma 1.5.1 *[23, 15.1] Let M be a module and N be a submodule of M. Then K.dim(M) is defined if and only if k.dim(N) and K.dim(M/N) are both defined in which case*

$$K.dim(M) = \max\{K.dim(N),\ K.dim(M/N)\}.$$

Lemma 1.5.2 *[47, 6.2.8] If the module M has Krull dimension then*

$$K.dimM \leq \sup\{k.dim(M/E) + 1 \mid E \text{ is an essential submodule of } M\}.$$

Lemma 1.5.3 *[23, 15.6] Let M be a nonzero module with Krull dimension and and $f : M \to M$ an injective endomorphism. Then*

$$K.dim(M) \geq K.dim(M/f(M)) + 1.$$

1.5.2 Global dimension

Let M be a module. An injective resolution of M is an exact sequence of modules and homomorphisms

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to \cdots$$

such that each I_i is injective. Injective resolutions exist since every module can be embedded in an injective module. The **injective dimension** of M is defined to be the shortest length of an injective resolution

$$0 \to M \to I_0 \to I_1 \to \cdots \to I_n \to I_n \to 0$$

of M. If no such n exists, we define the injective dimension of M to be ∞. By the injective version of Schanuel's Lemma [41, 5.40], the injective dimension of a module M is well defined.

The **left global dimension** of a ring R is defined to be the supremum of the injective dimensions of left R-modules. We denote the left global dimension of a ring R by $\mathrm{lgl.dim}(R)$. Right global dimension of a ring is defined similarly. Rings with global dimension zero are semisimple rings, that is rings R such that every left R-module is injective [40, 2.9]. If R has left global dimension one then it is called **left hereditary**.

Chapter 2

Injective hulls of simple modules

2.1 Motivation

The finiteness property which is at the core of this work has its roots in Krull's intersection theorem, which dates back to 1928. In [39], W. Krull proved the following:

Theorem 2.1.1 *Let* $(R, \mathfrak{m})$ *be a commutative local ring. Then* $\bigcap_{i=1}^{\infty} \mathfrak{m}^i = 0$.

Indeed, this means that the ring R is a Hausdorff space in the $\mathfrak{m}$-adic topology.

As a generalization of the Krull's intersection theorem, we have the famous **Jacobson's conjecture**. Jacobson asked in his book "Structure of Rings" [30], which is dated 1956, whether for a right Noetherian ring R with Jacobson radical $J(R)$ it is true that $\bigcap_{i=1}^{\infty} J^i(R) = 0$.

In 1965, Herstein answered the Jacobson conjecture in the negative by providing the following example in [27]: Let D be a commutative Noetherian domain with field of fractions Q and suppose that the Jacobson radical $J(D)$ of D is nonzero. Then we form the ring of 2×2 triangular matrices of the form

$$R = \left\{ \left(\begin{array}{cc} d & a \\ 0 & b \end{array} \right) \mid d \in D,\ a, b \in Q \right\}.$$

Then R is right Noetherian and its Jacobson radical $J(R)$ satisfies

$$J(R)^n \supset \left\{ \begin{pmatrix} 0 & a \\ 0 & 0 \end{pmatrix} \mid a \in Q \right\}$$

and so $\bigcap_{i=1}^{\infty} J(R)^i$ cannot be zero and R does not satisfy the Jacobson's conjecture.

This showed that if one assumes the ring R to be one sided Noetherian only, then the Jacobson's conjecture fails. Herstein then reformulated Jacobson's conjecture, asking the same question for two sided Noetherian rings.

Some classes of rings have been tested whether they satisfy Jacobson's conjecture. Among these classes there is the class of fully bounded Noetherian rings. A right Noetherian ring is said to be **right bounded** if every essential right ideal of it contains a two sided ideal. A ring R is called **right fully bounded right Noetherian** (right FBN-ring for short) if it is a right Noetherian ring whose prime factors R/P are right bounded for every prime ideal P. Left FBN rings are defined in the similar way. A **fully bounded Noetherian** ring (FBN ring for short) is a Noetherian ring which is both left and right fully bounded.

Jategaonkar showed in 1974 that FBN rings satisfy Jacobson's conjecture [33]. A key step in his proof is the following:

Proposition 2.1.2 *[33, Corollary 3.6] Over a FBN ring, any finitely generated module with essential socle has a composition series.*

We call a module M **locally Artinian** if all of its finitely generated submodules are Artinian. Observe that the above conclusion is equivalent to the property that injective hulls of simple modules over FBN rings are locally Artinian. Henceforth we will say that a ring R has property $(\diamond)$ or that it satisfies $(\diamond)$ condition if the injective hulls of simple R-modules are locally Artinian.

Jategaonkar then moves on to prove the Jacobson conjecture for FBN rings in [33, Theorem 3.7] in the following way. If $\{S_i \mid i \in I\}$ is a set of representatives of the isomorphism classes of simple R-modules, then the direct sum of the injective hulls $E(S_i)$, $i \in I$, is a faithful R-module. The crux of the proof is that the above observation for FBN rings implies that $\bigcap\{J^n(R) \mid n \in \mathbb{N}\}$ annihilates each $E(S_i)$, $i \in I$. Hence the intersection $\bigcap\{J^n(R) \mid n \in \mathbb{N}\}$ must be zero.

The natural question is then whether arbitrary Noetherian rings have property $(\diamond)$. However, this was shown not to be true by Musson (see [49] or [51]). In [51, Theorem 1], Musson constructed, for every positive integer n, a Noetherian prime ring R of Krull dimension $n+1$ with a finitely generated essential extension W of a simple R-module V such that

(i) W has Krull dimension n (hence it is not Artinian), and

(ii) W/V is n-critical and cannot be embedded in any of its proper submodules.

It should be noted that if R is a Noetherian ring which has property $(\diamond)$, then R satisfies the Jacobson's conjecture by the following argument.

Proposition 2.1.3 *If R is a Noetherian ring which has property $(\diamond)$ then R satisfies the Jacobson's conjecture.*

Proof: For all $0 \neq a \in R$, we can choose by Zorn's lemma a left ideal I_a of R maximal with respect to $a \notin I_a$. Then $\bigcap_{0 \neq a \in R} I_a = 0$ and $(Ra+I_a)/I_a \leq R/I_a$ is an essential extension of the simple left R-module $(Ra+I_a)/I_a$ for all $0 \neq a \in R$. By property $(\diamond)$, R/I_a has finite length and so there exists $i_a \geq 0$ such that $J^{i_a}(R/I_a) = 0$, *i.e.* $J^{i_a}(R) \subseteq I_a$, where J denotes the Jacobson radical of R. From this it follows that $\bigcap_{i=1}^{\infty} J^i \subseteq \bigcap_{a \in R} J^{i_a} \subseteq \bigcap I_a = 0$. $\square$

This makes property $(\diamond)$ interesting in its own and some Noetherian rings have been tested whether they have this property or not. We consider in the following two sections property $(\diamond)$ for some Noetherian rings.

2.2 Positive examples

2.2.1 Commutative Noetherian rings

Commutative Noetherian rings have property $(\diamond)$, as Matlis showed in 1960 that if R is a commutative Noetherian ring and A is an R-module, then the property that A being an essential extension of its socle is equivalent to, among other things, the property that every finitely generated submodule of A has finite length [45, Theorem 1].

2.2.2 FBN Rings, PI rings, and module finite algebras

More general than commutative Noetherian rings, as we mentioned above, FBN rings have property ($\diamond$). Some large classes of rings are indeed FBN rings. A **polynomial identity** on a ring R is a polynomial $p(x_1, x_2, \ldots, x_n)$ in noncommuting variables $x_1, x_2, \ldots, x_n$ with coefficients from $\mathbb{Z}$ such that $p(r_1, r_2, \ldots, r_n) = 0$ for all $r_1, r_1, \ldots, r_n \in R$. A **polynomial identity ring** or a **PI ring** for short, is a ring R which satisfies some monic polynomial identity. For example, a commutative ring is a PI ring since it satisfies the polynomial identity $p(x, y) = xy - yx$. Also, the Amitsur-Levitzki Theorem states that if A is a commutative ring then the matrix ring $M_n(A)$ is a PI ring [47, 13.3.3]. It is known that a Noetherian PI ring is a FBN ring and so Noetherian PI rings have property ($\diamond$).

More specifically, let R be an algebra over a commutative ring S. Then we can view R as an S-module. We say that R is a **module finite** S-algebra if R is a finitely generated S-module. Since $R \cong \mathrm{End}_R(R_R) \subseteq \mathrm{End}_S(R)$ as rings, then any polynomial identity satisfied in $\mathrm{End}_S(R)$ will also be satisfied in R. By the Amitsur-Levitzki Theorem, every matrix ring over a commutative ring is a PI ring, and so is every factor ring of a subring of such a ring. In particular, $\mathrm{End}_S(R)$ is a PI ring. From this one can conclude that a module finite algebra over a commutative ring is a PI ring and thus has property ($\diamond$).

In particular, the following two algebras are PI rings and they have property ($\diamond$). Let k be a field. The **coordinate ring of the quantum plane** is the k-algebra generated by the elements a, b subject to the relation $ab = qab$ is a PI ring when the parameter $q \in k$ is an nth root of unity. This is because in this case $k_q[a, b]$ is finitely generated over its center $k[a^n, b^n]$. Also, the **quantized Weyl algebra**, which is the k-algebra generated by the elements a, b subject to the relation $ab - qba = 1$ is also a PI ring when q is an nth root of unity, again being finitely generated over its center $k[a^n, b^n]$.

2.2.3 Dahlberg's $U(\mathfrak{sl}_2(\mathbb{C}))$ example and down-up algebras

Among other, noncommutative examples, Dahlberg [13] showed that the universal enveloping algebra $U(\mathfrak{sl}(2,\mathbb{C}))$ has property $(\diamond)$. This algebra is indeed a member of a larger class of algebras known as the down-up algebras, introduced by Benkart and Roby in [6].

Let k be a field. For fixed but arbitrary parameters $\alpha,\beta,\gamma \in k$ one defines the **down-up algebra** $A = A(\alpha,\beta,\gamma)$ as the k-algebra generated by the elements u and d subject to the relations

$$d^2u = \alpha udu + \beta ud^2 + \gamma d,$$
$$du^2 = \alpha udu + \beta u^2 d + \gamma u.$$

By [36], A is Noetherian if and only if it is a domain, if and only if $\beta \neq 0$. Recently, the full characterization of down-up algebras which have property $(\diamond)$ has been obtained by Carvalho, Lomp, and Pusat-Yilmaz [10], Carvalho and Musson [9], and Musson [50].

Proposition 2.2.1 *[10, 9, 50] Let $A = A(\alpha,\beta,\gamma)$ be a Noetherian down-up algebra over a field k of characteristic zero. Then A has property $(\diamond)$ if and only if the roots of $X^2-\alpha X-\beta$ are roots of unity.*

In the general case of a noncommutative Noetherian ring, Carvalho, Lomp, and Pusat-Yilmaz proved the following using a Krull dimension argument.

Lemma 2.2.2 *[10, (1.4)] A semiprime Noetherian ring of Krull dimension one has property $(\diamond)$.*

This result in particular means that the first Weyl algebra $A_1(\mathbb{C}) = \mathbb{C}[x][y;\partial/\partial x]$ has property $(\diamond)$. On the other hand, they also obtained a reduction of the problem when a Noetherian algebra A has a nontrivial centre.

Proposition 2.2.3 *[10, (1.6)] The following statements are equivalent for a countably generated Noetherian algebra A with Noetherian center over an algebraically closed uncountable field K.*

(a) Injective hulls of simple left A-modules are locally Artinian;

(b) Injective hulls of simple left $A/\mathfrak{m}A$-modules are locally Artinian for all maximal ideals $\mathfrak{m}$ of the center $Z(A)$ of A.

The above result applies to the three dimensional Heisenberg Lie algebra $\mathfrak{h}$ over $\mathbb{C}$ which is generated by x, y, z with the Lie algebra structure is given by $[x, y] = z$ and $[x, z] = 0 = [y, z]$. Let $A = U(\mathfrak{h})$ be the universal enveloping algebra of $\mathfrak{h}$. The center of A is $\mathbb{C}[z]$ and the maximal ideals of the center are of the form $\langle z - \lambda \rangle$ where $\lambda \in \mathbb{C}$. Then for a maximal ideal $\mathfrak{m}$ of the centre $Z(A)$, the factor $A/\mathfrak{m}A$ is either $\mathbb{C}[x, y]$, which is a commutative Noetherian domain, or is the first Weyl algebra. In both cases these factors have property ($\diamond$) and so does the Heisenberg Lie algebra [10, (1.7)].

The results of Carvalho, Lomp, and Pusat-Yilmaz also apply to certain quantum groups, as we show in the following examples.

2.2.3.1 The quantized enveloping algebra $U_q(\mathfrak{sl}_2)$

We fix a ground field k and an element $q \in k$ with $q \neq 0$ and $q^2 \neq 1$. The quantized enveloping algebra $U = U_q(\mathfrak{sl}_2)$ is the k-algebra generated by E, F, K, K^{-1} subject to the relations

$$KK^{-1} = 1 = K^{-1}K, \quad KEK^{-1} = q^2E, \quad KFK^{-1} = q^{-2}F, \quad EF - FE = \tfrac{K - K^{-1}}{q - q^{-1}}$$

First assume that q is not a root of unity. The element $C = EF + (q^{-1}K + qK^{-1})/(q - q^{-1})^2$ is a central element of $U_q(\mathfrak{sl}_2)$. Indeed, the center of $U_q(\mathfrak{sl}_2)$ is the subalgebra $k[C]$ (see [32, Proposition 2.18])

We will use Proposition 2.2.3 to conclude that $U_q(\mathfrak{sl}_2)$ has property ($\diamond$). Consider the maximal ideals of the center $k[C]$, which are of the form $\langle C - \lambda \rangle$ for $\lambda \in k$. By [53, Theorem 1], the ideal $\langle C - \lambda \rangle$ is a completely prime ideal of U for every $\lambda \in k$ (although it is proved over the complex numbers, the proof works for an arbitrary algebraically closed field k of characteristic zero). Also, Jordan shows in [35] that $U_q(\mathfrak{sl}_2)$ has Krull dimension 2, provided q is not a root of unity. Hence the factor $U_q(\mathfrak{sl}_2)/\langle C-\lambda\rangle U_q(\mathfrak{sl}_2)$ is a primitive ring of Krull dimension 1. Then by Lemma 2.2.2, each such factor has property $(\diamond)$. This implies in turn that U has property $(\diamond)$ by the above proposition.

In the case q is a root of unity, then $U_q(\mathfrak{sl}_2)$ is a PI ring ([8, III.6.2.]) and has property $(\diamond)$.

2.2.3.2 The algebra $U_q^+(\mathfrak{sl}_n)$

We first consider the algebra $U = U_q^+(\mathfrak{sl}_3)$, which is the k-algebra with generators e_1, e_2 subject to the Serre relations (see [8, I.6.2.])

$$\begin{aligned} E_1^2E_2-(q+q^{-1})E_1E_2E_1+E_2E_1^2 &= 0 \\ E_2^2E_1-(q+q^{-1})E_2E_1E_2+E_1E_2^2 &= 0 \end{aligned}$$

Thus $U_q^+(\mathfrak{sl}_3)$ can be realized as the down-up algebra $A = A(q+q^{-1},-1,0)$. By Proposition 2.2.1, A has property $(\diamond)$ if and only if the roots of the polynomial $X^2-(q+q^{-1})X+1$ are roots of unity. It is easy to see that the roots of this polynomial are q, q^{-1}. Hence it follows that the algebra $U_q^+(\mathfrak{sl}_3)$ has property $(\diamond)$ if and only if q is a root of unity.

Now we consider the general case. Let $U = U_q^+(\mathfrak{sl}_n)$ be the algebra with generators $E_1, \ldots, E_{n-1}$ and relations

$$\begin{aligned} E_iE_j = E_jE_i, &\qquad |i-j| \geq 2, \\ E_i^2E_j-(q+q^{-1})E_iE_jE_i+E_jE_i^2 = 0, &\qquad |i-j| = 1. \end{aligned}$$

It is easy to see that $U_q^+(\mathfrak{sl}_3)$ is the homomorphic image of U: the map which is defined by $E_i \mapsto E_i$ for $i = 1, 2$ and $E_i \mapsto 0$ for $i = 3, \ldots, n-1$ gives rise to an epimorphism $U_q^+(\mathfrak{sl}_n) \twoheadrightarrow U_q^+(\mathfrak{sl}_3)$. Since property $(\diamond)$ is inherited by factor rings, we conclude that U has property $(\diamond)$ if and only if q is a root of unity.

2.2.3.3 Quantum affine n-space

Let k be a field and q be an element of k. Let $A = k_q\langle x, y\rangle$ denote the coordinate ring of the quantum plane. If q is a root of unity, then A is a PI ring and has property $(\diamond)$.

Carvalho and Musson showed in [9] that if q is not a root of unity then A does not have property ($\diamond$).

Quantum affine n-space is the algebra $\mathcal{O}_q(k^n)$ with generators $x_1, \ldots, x_n$ and relations

$$x_i x_j = q x_j x_i$$

for all $i < j$. For any $1 \leq i < j \leq n$, the assignment which sends x_l to x_l if $l = i, j$ and to zero otherwise gives rise to an epimorphism from $\mathcal{O}_q(k^n)$ to the coordinate ring of the quantum plane A. Hence $\mathcal{O}_q(k^n)$ does not have property ($\diamond$) if q is not a root of unity.

Indeed, this argument also works for the multiparameter quantum affine space. If $\mathbf{q} \in M_n(k^\times)$ is a multiplicatively antisymmetric matrix, the corresponding **multiparameter quantum affine space** is the k-algebra $\mathcal{O}_\mathbf{q}(k^n)$ with generators $x_1, \ldots, x_n$ and subject to the relations

$$x_i x_j = q_{ij} x_j x_i$$

for all i, j. Then, in the same way, there is an epimorphism from $\mathcal{O}_\mathbf{q}(k^n)$ to any of the algebras with generators x_i, x_j and relation $x_i x_j = q_{ij} x_j x_i$. Hence the multiparameter quantum affine space does not have property ($\diamond$) if the entries of the matrix $\mathbf{q}$ are not all roots of unity.

2.2.4 Group rings

In the case of group rings, it has been shown that $\mathbb{Z}G$ and kG where k is a field which is algebraic over a finite field and G is polycyclic-by-finite both have property ($\diamond$) by the works of Jategaonkar [34] and Roseblade [57].

2.3 Negative examples

The list of rings which do not have property ($\diamond$) includes the following rings.

We already noted that the coordinate ring of the quantum plane and the quantized Weyl algebra do not have property ($\diamond$) when the parameter $q \in k$ is not a root of unity [9].

In the case of group rings, Musson showed that if k is a field which is not algebraic over a finite field and G is polycyclic-by-finite which is not nilpotent-by-finite, then kG does not have property ($\diamond$) [49].

2.3.1 Goodearl and Schofield's example

Goodearl and Schofield [21] showed that there exists a nonprime Noetherian ring of Krull dimension one which does not have property ($\diamond$). They start with a skew field extension $F \subset E$ such that $\dim_F E < \infty$ while E_F is transcendental. Then the ring

$$R = \begin{pmatrix} E[x] & 0 \\ E[x] & F[x] \end{pmatrix}$$

of triangular matrices is a nonprime Noetherian ring of Krull dimension one which has a simple module with a non-Artinian cyclic essential extension.

2.3.2 Musson's example

We already mentioned Musson's example in § 2.1. Here we consider his construction in more detail. Let k be a field of characteristic zero and L be a vector space over k with basis $y, x_0, x_1, \ldots, x_{n-1}$. We define a Lie bracket on L in the following way:

$$[x_i, x_j] = 0, \quad [x_0, y] = x_0$$
$$[x_i, y] = x_i + x_{i-1} \quad \text{for } i = 1, 2, \ldots, n-1.$$

Then we let R to be the enveloping algebra of L. Then R is a prime Noetherian ring of Krull dimension $n+1$ which does not have property ($\diamond$). In the particular case of $n = 1$, L has the form

$$L = kx_0 \oplus ky, \quad \text{where } [x_0, y] = x_0.$$

In this case, if k is algebraically closed then it follows from [7, p. 71] that L is an epimorphic image of any finite dimensional solvable Lie algebra which is not nilpotent. Hence, the enveloping algebra of a finite dimensional solvable but not nilpotent Lie algebra over an algebraically closed field does not have property ($\diamond$) [51, Theorem 2].

2.3.3 Stafford's result

One of the corner stones of our work on both Lie superalgebras and on differential operator rings is Stafford's result on Weyl algebras. Let A_n be the nth Weyl algebra over the complex numbers. In general, a simple module M over a Noetherian ring R with finite Gelfand-Kirillov dimension is called **holonomic** if $GK\dim M = \frac{1}{2}GK\dim(R/\mathrm{Ann}M)$. Stafford studies nonholonomic modules over Weyl algebras and enveloping algebras in [63] and in particular he answers a question of Björk in the negative which asks whether every simple A_n-module is holonomic. This is done by constructing explicitly a simple A_n-module which has Gelfand-Kirillov dimension $2n-1$ in the following main result of his paper.

Theorem 2.3.1 *[63, Theorem 1.1] For $2 \le i \le n$ pick $\lambda_i \in \mathbb{C}$ that are linearly independent over $\mathbb{Q}$. Then the element*

$$\alpha = x_1 + y_1\left(\sum_2^n \lambda_i x_i y_i\right) + \sum_2^n (x_i + y_i)$$

generates a maximal right ideal of A_n. In particular, the simple A_n-module $A_n/\alpha A_n$ has Gelfand-Kirillov dimension $2n-1$ and projective dimension one.

As a corollary of the above theorem Stafford gives the following.

Corollary 2.3.2 *[63, Corollary 1.4] Let $\alpha \in A_n$ be as in the theorem. Then $A_n/x_1\alpha A_n$ is an essential extension of the simple A_n-module $A_n/\alpha A_n$ by the module A_n/x_1A_n, which has Krull dimension $n-1$.*

This means that for all $n \geq 2$, the Weyl algebra A_n has a simple module which has a cyclic essential extension of Krull dimension $n-1$. Since the Artinian modules are exactly the ones with Krull dimension zero, this means that the Weyl algebra A_n does not have property $(\diamond)$ for $n \geq 2$. This result will be central when we study property $(\diamond)$ for nilpotent Lie superalgebras and differential operator rings.

2.4 Related works in the literature

2.4.1 V-rings and injective hulls of modules of finite length

Michler and Vilamayor studied in [48] rings over which every simple left module is injective. Such rings are called **left V-rings** after Vilamayor. Right V-rings are defined similarly. Of course, V-rings have property ($\diamond$). In the commutative case, a result of Kaplansky states that a commutative ring R is von Neumann regular if and only if R is a V-ring. In particular, V-rings have zero Jacobson radical.

2.4.2 The works of Hirano, Jans, Vámos, and Rosenberg and Zelinsky

Rosenberg and Zelinsky considered in [58] the rings R with the property that every simple module has an injective hull of finite length. Obviously these rings have property ($\diamond$). The main problem of their work was to study the question whether a module of finite length has an injective hull of finite length.

In Section 3 of their paper, they drop any finiteness condition on the ring R and they assume that simple left R-modules have an injective hull of finite length. They show in [58, Theorem 4] that such a ring satisfies the Jacobson's conjecture. In [58, Theorem 5] they show that for a commutative ring R with unit, the injective hull of a simple R-module R/M has finite length if and only if the localization R_M is a ring with minimum condition.

Another similar finiteness property is the so called co-Noetherian property. Seeking a dual notion of finitely generated, Vámos introduced the notion of finitely embedded in [66]. There, an R-module M is called **finitely embedded (f.e.)** if its injective hull $E(M)$ is a finite direct sum of injective hulls of simple modules. He then showed that this is equivalent to saying that M has a finitely generated essential socle [66, Lemma 1]. Vámos obtained the duals of a number of results on finitely generated modules. In particular, in connection with the characterization of Noetherian modules as the ones having every submodule finitely generated, he showed that a module M is Artinian if and only if every factor module of M is finitely embedded [66, Proposition 2*]. Another remarkable result from the same paper is the following:

Theorem 2.4.1 *[66, Theorem 2] For a commutative ring R the following conditions are equivalent:*

(i) Every injective hull of a simple module is Artinian;

(ii) The localization R_M is Noetherian for every maximal ideal M of R.

In [31] rings over which injective hulls of simple modules are Artinian are called left **co-Noetherian**. Jans also shows that if a ring R is left Noetherian and left co-Noetherian then it satisfies the Jacobson's conjecture [31, Theorem 2.1], which also follows from Proposition 2.1.3 in the more relaxed case.

Observe that rings with the property that every simple module has an injective module of finite length are co-Noetherian, but there are co-Noetherian rings which fail to have this property. Jans provides an example of such a ring: the ring of integers is co-Noetherian and although the injective hulls for simple $\mathbb{Z}$-modules satisfy the minimum condition, they do not have composition series.

A characterization of the rings considered by Rosenberg and Zelinsky is given by Hirano in [28]. Hirano shows for a ring R, that the injective hulls of simple left R-modules have finite length if and only if for every left R-module M, the intersection of all submodules N with M/N has finite length is zero [28, Theorem 1.1]. He also shows that if the injective hulls of simple left R-modules have finite length or if R is co-Noetherian, then any finite normalizing extension of R has the same property [28, Theorem 1.8, Theorem 2.2].

In the noncommutative case, Hirano conjectures that if R is left co-Noetherian such that every primitive factor of $R[x]$ is Artinian, then $R[x]$ is also a left co-Noetherian.

2.4.3 Donkin's work

In [16], Donkin considers locally finite dimensional modules over group rings. If G is a polycyclic-by-finite group and k is a field of characteristic zero, for a finite dimensional kG-module V he proves that (i) any essential extension of V is Artinian, and (ii) the endomorphism ring of the injective hull $E(V)$ is Noetherian. In positive characteristic this has been shown by Musson in [49].

In the general case of a Hopf algebra over a field k of characteristic zero, Donkin proves two results corresponding to (i) and (ii) above. Namely, he proves the following:

(i) Let H be an affine Hopf algebra over a field k of characteristic zero. For any finite dimensional H-comodule V, the endomorphism ring $\mathrm{End}_H(E(V))$ of the injective hull (as a comodule) $E(V)$ of V is Noetherian [17, theorem A].

(ii) Let H be an affine Hopf algebra over a field k of characteristic zero. Then any essential H-comodule extension of a finite dimensional H-comodule is Artinian [17, Theorem B].

He also shows in the last section of [17] that the Noetherian rings $\mathrm{End}_H(E(V))$ of (i) satisfy the Jacobson's conjecture.

Applications of these results can be found in representation theory of algebraic groups, of polycyclic groups and of Lie algebras. In particular, an application to algebraic groups is given in [17, Corollary 6.4]. In [18], Donkin considers the applications of the above results to enveloping algebras.

2.4.4 Injective hulls of Iwasawa algebras

Let G be a compact p-adic Lie group. The Iwasawa algebra with coefficients in some finite integral ring extension $\mathcal{O}$ of $\mathbb{Z}_p$ is defined to be

$$\mathcal{O}G := \varprojlim \mathcal{O}[G/U]$$

where the inverse limit is taken over all the open normal subgroups U of G. We write KG for the tensor product $K \otimes_{\mathcal{O}} \mathcal{O}G$, where K is the field of fractions of $\mathcal{O}$.

Nelson explicitly computes and makes use of the injective hull of the trivial module in his paper [52] to prove the following result:

Theorem 2.4.2 *Let G be a uniform nilpotent pro-p group and K a finite extension of $\mathbb{Q}_p$. Then any primitive ideal P of KG such that KG/P is finite dimensional is localisable.*

Note that the primitive ideals of the above theorem can be thought of as kernels of finite dimensional irreducible representations of KG. More precisely, Nelson uses the injective hulls $E(KG/P)$ of KG-modules first when $KG/P \cong K$ is the trivial module and then when KG/P is a general finite dimensional module. The injective hull of the trivial module is computed to be a form of polynomial ring [52, Theorem 3.6] which in the general case also acts as a base for the computation of the injective hull of a finite dimensional module [52, Theorem 4.2].

Chapter 3

Modules over nilpotent Lie superalgebras

3.1 Introduction

Let k be an algebraically closed field of characteristic zero and let $\mathfrak{g}$ be a finite dimensional Lie algebra over k which is solvable but not nilpotent. Musson's result which we have presented in § 2.3.2 shows that the enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ does not have property $(\diamond)$. Then it is natural to ask for which finite dimensional nilpotent Lie algebras $\mathfrak{g}$ over k the enveloping algebra $U(\mathfrak{g})$ has property $(\diamond)$. We address this question in this chapter, and give a complete answer in a slightly more general context of Lie superalgebras. Namely we prove the following main result of this chapter. Recall that a **central abelian direct factor** of a Lie algebra $\mathfrak{g}$ is an abelian Lie subalgebra $\mathfrak{a}$ of $\mathfrak{g}$ such that $\mathfrak{g} = \mathfrak{h} \times \mathfrak{a}$ for some Lie subalgebra $\mathfrak{h}$ of $\mathfrak{g}$.

Main Theorem 3.1.1 *Let k be an algebraically closed field of characteristic zero. The following statements are equivalent for a finite dimensional nilpotent Lie superalgebra $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ over k.*

(a) Finitely generated essential extensions of simple $U(\mathfrak{g})$-modules are Artinian.

(b) Finitely generated essential extensions of simple $U(\mathfrak{g}_0)$-modules are Artinian.

(c) $ind(\mathfrak{g}_0) \geq \dim(\mathfrak{g}_0) - 2$, where $ind(\mathfrak{g}_0)$ denotes the index of $\mathfrak{g}_0$.

(d) Up to a central abelian direct factor $\mathfrak{g}_0$ is isomorphic to one of the following

(i) a nilpotent Lie algebra with abelian ideal of codimension 1;

(ii) the 5-dimensional Lie algebra $\mathfrak{h}_5$ with basis $\{e_1, e_2, e_3, e_4, e_5\}$ and nonzero brackets given by

$$[e_1, e_2] = e_3, \quad [e_1, e_3] = e_4, \quad [e_2, e_3] = e_5;$$

(iii) the 6-dimensional Lie algebra $\mathfrak{h}_6$ with basis $\{e_1, e_2, e_3, e_4, e_5, e_6\}$ and nonzero brackets given by

$$[e_1, e_3] = e_4, \quad [e_2, e_3] = e_5, \quad [e_1, e_2] = e_6.$$

This together with Musson's example gives a characterization of finite dimensional solvable Lie algebras $\mathfrak{g}$ whose enveloping algebra $U(\mathfrak{g})$ has property ($\diamond$).

Corollary 3.1.2 *Let $\mathfrak{g}$ be a finite dimensional solvable Lie algebra over an algebraically closed field of characteristic zero. The enveloping algebra $U(\mathfrak{g})$ of $\mathfrak{g}$ has property ($\diamond$) if and only if $\mathfrak{g}$ is nilpotent and is isomorphic up to an abelian direct factor to a Lie algebra with an abelian ideal of codimension 1 or to $\mathfrak{h}_5$ or to $\mathfrak{h}_6$.*

The proof of the main theorem will consist of several steps. Stafford's result plays a central role in our work and we first reformulate and prove his theorem on Weyl algebras for more general fields. Also we depend on a kind of Artin-Rees property for finitely generated essential extensions of simple modules and on the primitive factors to have property ($\diamond$). More specifically, we first show that, Noetherian rings whose primitive ideals contain nonzero ideals with a normalizing sequence of generators have property ($\diamond$), provided that their primitive factors have property ($\diamond$).

The next step is to examine the primitive ideals of the enveloping algebra of a finite dimensional nilpotent Lie superalgebra. We show that for such a Lie superalgebra $\mathfrak{g}$, the ideals of the universal enveloping algebra $U(\mathfrak{g})$ contain supercentralizing sequence of generators. This result together with the first step shifts our problem to a study of the primitive factors of $U(\mathfrak{g})$.

The primitive factors of the enveloping algebra $U(\mathfrak{g})$ are given by Bell and Musson as tensor products of the form $\mathrm{Cliff}_q(k)\otimes A_p(k)$. Clifford algebras are finite dimensional algebras and hence they have property $(\diamond)$. We show that if A is a k-algebra, then the tensor product $\mathrm{Cliff}_q(k)\otimes A$ has property $(\diamond)$ for all (for some) q if and only if A has property $(\diamond)$. So that in our case it is enough to consider the Weyl algebras. Since we already know that the only Weyl algebra having property $(\diamond)$ is the first Weyl algebra, it remains to know what controls the order of the Weyl algebra appearing in the primitive factors. A result by Herscovich shows that this is controlled by the so called index of the underlying even part $\mathfrak{g}_0$ of $\mathfrak{g}$. In our case this imposes the condition $\mathrm{ind}(\mathfrak{g}_0)\geq \dim(\mathfrak{g}_0)-2$.

In the last step we list all finite dimensional nilpotent Lie algebras $\mathfrak{g}$ which satisfies the formula $\mathrm{ind}(\mathfrak{g})\geq \dim(\mathfrak{g})-2$.

3.2 Stafford's result over algebraically closed fields of characteristic zero

We have seen Stafford's result in Section 2.3.3, which says that the only complex Weyl algebra which has property $(\diamond)$ is the first Weyl algebra. While Stafford proved his result over the field of complex numbers, we show in this section that Stafford's results for the nth Weyl algebra are valid for an arbitrary field k which is at least $n-1$ dimensional over the rationals.

We first prove some general observations which will be required in the proof of the main result.

Lemma 3.2.1 *Let $A_1(k)$ be the first Weyl algebra over a field k with generators x and y. Then for any $m,n\geq 0$ we have*

$$[xy, x^n y^m] = (m-n)x^n y^m$$

Proof: First note that the equations $yx^n = x^n y - nx^{n-1}$ and $y^m x = xy^m - my^{m-1}$ hold in A_1. By direct computation we see that

$$\begin{aligned} [xy, x^n y^m] &= xyx^n y^m - x^n y^m xy \\ &= x(x^n y - nx^{n-1})y^m - x^n(xy^m - my^{m-1})y \\ &= (m-n)x^n y^m. \end{aligned}$$

□

Lemma 3.2.2 *Let k be a field which is at least $n-1$-dimensional over $\mathbb{Q}$, A_n be the nth Weyl algebra over k with generators $x_1, \ldots, x_n, y_1, \ldots, y_n$ and let $S = k[x_2, \ldots, x_n, y_2, \ldots, y_n]$. Given $s \in S$ and $\lambda_2, \ldots, \lambda_n \in k$ which are linearly independent over $\mathbb{Q}$, we define $\theta(s) = \sum_{i=2}^n \lambda_i [x_i y_i, s] - ps$ where $p = \sum_{i=2}^n \lambda_i u_i$ for some $u_i \in \mathbb{Z}$. If $s = x_2^{v_2} \ldots x_n^{v_n} y_2^{w_2} \ldots y_n^{w_n}$, then $\theta(s) = \mu s$ for some $\mu \in k$. Furthermore, $\theta(s) = 0$ if and only if $w_i - v_i = u_i$ for $2 \le i \le n$.*

Proof: By Lemma 3.2.1, it follows that

$$\begin{aligned} \theta(s) &= \sum_{i=2}^n \lambda_i [x_i y_i, s] - \sum_{i=2}^n \lambda_i u_i s = \sum_{i=2}^n \lambda_i (w_i - v_i)s - \sum_{i=2}^n \lambda_i u_i s \\ &= \sum_{i=2}^n \lambda_i (w_i - v_i - u_i)s \end{aligned}$$

Since the λ_i are linearly independent over $\mathbb{Q}$ we have $\theta(s) = 0$ if and only if $w_i - v_i - u_i = 0$ for each i, and this completes the proof. □

The proof of our next result is rather long, we therefore divide it into several steps in order to make it easier to follow. A proof of the case $n = 2$ can also be found in [38, Proposition 8.8].

Theorem 3.2.3 *Let k and A_n be as in the preceding lemma. For $2 \le i \le n$ pick $\lambda_i \in k$ that are linearly independent over $\mathbb{Q}$. Then the element*

$$\alpha = x_1 + y_1 \left(\sum_{i=2}^n \lambda_i x_i y_i \right) + \sum_{i=2}^n (x_i + y_i)$$

generates a maximal right ideal of A_n.

Proof: The monomials of the form $\gamma = x_1^{a_1}\dots x_n^{a_n} y_2^{b_2}\dots y_n^{b_n} y_1^{b_1} \in A_n$, where $a_i, b_i \geq 0$, form a basis for A_n as a k-vector space. We define the degree of a monomial γ to be the $2n$-tuple $(a_1,\dots,a_n,b_2,\dots,b_n,b_1)$. We order these $2n$-tuples lexicographically, meaning that $(a_1,\dots,a_{2n}) < (b_1,\dots,b_{2n})$ if and only if there exists $1 \leq j \leq 2n$ such that $a_j < b_j$ and $a_i = b_i$ for all $i < j$. Suppose that our claim is false and there exists $\beta \in A_n \backslash \alpha A_n$ such that $\alpha A_n + \beta A_n \neq A_n$. For the rest of the proof we fix an element β satisfying these properties but of the smallest possible degree.

Step 1. We first note that $\beta \notin k[y_1]$. Otherwise, suppose that $\beta \in k[y_1]$, say $\beta = p(y_1) \in k[y_1]$. First note that $[\alpha,\beta] \notin \alpha A_n$, simply because while $[\alpha,\beta] = \frac{\partial p(y_1)}{\partial y_1}$ has x_1-degree zero, every nonzero element of αA_n has positive x_1-degree. Thus, $[\alpha,\beta]$ is certainly an element which is of smaller degree than that of β satisfying $\alpha A_n + (\alpha\beta - \beta\alpha)A_n \subset \alpha A_n + \beta A_n \neq A_n$, contradicting the minimality assumption on β.

We now proceed to show that the above observation places strong restraints on the commutator $[\alpha,\beta]$. Let us write $R = A_{n-1}[y_1]$, where A_{n-1} is the Weyl algebra of order $n-1$ with generators x_i, y_i with $2 \leq i \leq n$.

Step 2. Now we show that $\beta \in R$. Note that α is monic of degree 1 as a polynomial in x_1, with coefficients in R. Hence, if we write $\beta = x_1\beta_1 + \beta_2$ for some $0 \neq \beta_1 \in A_n$ and $\beta_2 \in R$, then $\gamma = \beta - \alpha\beta_1$ still satisfies the same properties with β but it has smaller degree. The minimality of $\deg\beta$ implies $\beta_1 = 0$ and hence $\beta \in R$.

Thus, we know that $\beta \in R$, with degree, say, $(0,r_2,\dots,r_n,s_2,\dots,s_n,s_1)$. Write $\beta = \beta_1 + \beta_2$ where $\beta_1 = x_2^{r_2}\dots x_n^{r_n} y_2^{s_2}\dots y_n^{s_n} f$ for some $f \in k[y_1]$ and β_2 has degree less than $(0,r_2,\dots,r_n,s_2,\dots,s_n,0)$. Since $\beta \notin k[y_1]$ as we showed above, it follows that there exists at least one $2 \leq i \leq n$ for which r_i or s_i is nonzero, and therefore $\beta_1 \neq 0$.

Step 3. We claim that $[\alpha,\beta] = \beta y_1 \sum_{i=2}^n \lambda_i u_i$ for some $u_i \in \mathbb{Z}$. Consider the element

$$\gamma = \alpha\beta - \beta\alpha + \beta y_1 \sum_{i=2}^n \lambda_i (r_i - s_i).$$

We show that γ is zero and prove our claim. We substitute $\beta = \beta_1 + \beta_2$ in the expression for γ and write $\gamma = \gamma_1 + \gamma_2$ where $\gamma_j = \alpha\beta_j - \beta_j\alpha + \beta_j y_1 \sum_{i=2}^n \lambda_i (r_i - s_i)$. We claim that $\deg\gamma < \deg\beta$. Since for $i \geq 2$, the generators x_i, y_i commute with x_1 and y_1, it follows from Lemma 3.2.1 that

$$[y_1 x_i y_i, x_i^a y_i^b] = (b-a) y_1 x_i^a y_i^b. \tag{3.1}$$

Further, for any $r \in R$, we have $[x_i, r] = \frac{\partial}{\partial y_i}(r)$ for $1 \le i \le n$ and $[y_i, r] = -\frac{\partial}{\partial x_i}(r)$ for $2 \le i \le n$. Combining these observations shows that for any $r \in R$,

$$\begin{aligned} \alpha r - r\alpha &= (x_1 + y_1\left(\sum_{i=2}^n \lambda_i x_i y_i\right) + \sum_{i=2}^n (x_i + y_i))r - r(x_1 + y_1\left(\sum_{i=2}^n \lambda_i x_i y_i\right) + \sum_{i=2}^n (x_i + y_i)) \\ &= [x_1, r] + \sum_{i=2}^n [\lambda_i y_1 x_i y_i, r] + \sum_{i=2}^n [x_i + y_i, r] \end{aligned}$$

where each summand is of degree less than or equal to $\deg y_1 r$, and so $\deg(\alpha r - r\alpha) \le \deg y_1 r$. In particular,

$$\deg \gamma_2 \le \deg y_1 \beta_2 < (0, r_2, \ldots, s_n, 0) \le \deg \beta. \tag{3.2}$$

We now consider γ_1. By Equation 3.1 we have

$$[y_1 \sum_{i=2}^n \lambda_i x_i y_i, \beta_1] = y_1 \beta_1 \sum_{i=2}^n \lambda_i (s_i - r_i)$$

and this cancels with the last term in the expression for γ. Thus,

$$\gamma_1 = \alpha\beta_1 - \beta_1\alpha + \beta_1 y_1 \sum_{i=2}^n \lambda_i (r_i - s_i) \;=\; [x_1, \beta_1] + \sum_{i=2}^n [x_i + y_i, \beta_1].$$

and so $\deg \gamma_1 < \deg \beta_1$. Combined with Equation 3.2, this gives $\deg \gamma < \deg \beta$. Since $\gamma A_n + \alpha A_n \subseteq \beta A_n + \alpha A_n \neq A_n$, the minimality of $\deg \beta$ forces $\gamma \in \alpha A_n$. However, $\gamma \in R$ yet $\alpha A_n \cap R = 0$; and so $\gamma = 0$. This completes the proof of the claim, and hence we have $\alpha\beta - \beta\alpha = \beta y_1 \sum_{i=2}^n \lambda_i u_i$ for some $u_i \in \mathbb{Z}$.

Now set $S = k[x_2, \ldots, x_n, y_2, \ldots, y_n]$. We will sometimes write the monomial $x_2^{c_2} \ldots x_n^{c_n} y_2^{d_2} \ldots y_n^{d_n} \in S$ as z^γ for $\gamma = (c_2, \ldots, d_n)$. Again, we define a degree function on S, so that the monomial z^γ has degree $(c_2, \ldots, d_n)$ where these $(2n-2)$-tuples are ordered lexicographically.

Set $p = \sum_{i=2}^n \lambda_i u_i$ where the u_i are as defined in Step 3. Then we have the equation

$$\alpha\beta - \beta\alpha - p\beta y_1 = 0. \tag{3.3}$$

Write $\beta = \sum_{i=0}^{t} y_1^i b_i$ for $b_i \in S$ with $b_t \neq 0$. The following steps show that this equation leads to a contradiction. We do this by computing the first few b_i.

Step 4. We first expand Equation 3.3 by substituting $\beta = \sum_{i=0}^{t} y_1^i b_i$ and by computing first

$$\begin{aligned}\alpha\beta &= x_1 \sum_{i=0}^{t} y_1^i b_i + y_1 \left(\sum_{i=2}^{n} \lambda_i x_i y_i \right) \sum_{i=0}^{t} y_1^i b_i + \sum_{i=2}^{n} (x_i + y_i) \sum_{i=0}^{t} y_1^i b_i \\ &= \sum_{i=0}^{t} x_1 y_1^i b_i + \sum_{i=0}^{t} \sum_{j=2}^{n} y_1^{i+1} \lambda_j x_j y_j b_i + \sum_{i=0}^{t} \sum_{j=2}^{n} y_1^i (x_j + y_j) b_i\end{aligned}$$

and then

$$\beta\alpha = \sum_{i=0}^{t} y_1^i x_1 b_i + \sum_{i=0}^{t} \sum_{j=2}^{n} y_1^{i+1} b_i \lambda_j x_j y_j + \sum_{i=0}^{t} \sum_{j=2}^{n} y_1^i b_i (x_j + y_j)$$

and hence, Equation 3.3 in expanded form is

$$\sum_{i=2}^{t} i y_1^{i-1} b_i + \sum_{i=0}^{t} \sum_{j=2}^{n} y_1^{i+1} \lambda_j [x_j y_j, b_i] + \sum_{i=0}^{t} \sum_{j=2}^{n} y_1^i [x_j + y_j, b_i] - p \sum_{i=0}^{t} y_1^{i+1} b_i = 0. \tag{3.4}$$

Equating the coefficients of y_1^{t+1} gives

$$0 = \sum_{j=2}^{n} \lambda_j [x_j y_j, b_t] - p b_t. \tag{3.5}$$

Step 5. In the notation of Lemma 3.2.2, it follows from Step 4 that $\theta(b_t) = 0$. Hence, if we write $b_t = \sum b_{t_\gamma} z^\gamma$ for $\gamma = (c_2, \ldots, c_n, d_2, \ldots, d_n)$ then it follows from Lemma 3.2.2 that

$$b_t = \sum b_{t_\gamma} x_2^{c_2} \ldots x_n^{c_n} y_2^{c_2+u_2} \ldots y_n^{c_n+u_n}$$

for some $b_{t_\gamma} \in k$. We claim that this implies $t > 0$. Suppose $t = 0$. Then the coefficient of y_1^0 in Equation 3.4 gives $0 = \sum_{j=2}^{n} [x_j + y_j, b_t]$. Equivalently,

$$0 = -\sum_{\gamma} \sum_{i} b_{t_\gamma} c_i x_2^{c_2} \ldots x_i^{c_i-1} \ldots x_n^{c_n} y_2^{c_2+u_2} \ldots y_n^{c_n+u_n} + \sum_{\gamma} \sum_{i} b_{t_\gamma} (c_i + u_i) x_2^{c_2} \ldots y_i^{c_i+u_i-1} \ldots y_n^{c_n+u_n}. \tag{3.6}$$

We show that this forces $b_t \in k$. If $b_{t_\gamma} \neq 0$ for some $\gamma \neq (0)$, then for Equation 3.6 to still hold, two or more terms in Equation 3.6 must cancel and so these terms will certainly have the same degree. This implies either

$$(c_2, \ldots, c_{i-1}, \ldots, c_n, c_2 + u_2, \ldots, c_n + u_n) = (c_2', \ldots, c_n', c_2' + u_2, \ldots, c_j' + u_j - 1, \ldots, c_n' + u_n)$$

for some i and j, or one of two other similar equations should hold. It is clear that no such equation is possible, and this implies that $b_{t_\gamma} = 0$ whenever $(c_2,\dots,c_{i-1},\dots,c_n,c_2+u_2,\dots,c_n+u_n) \neq (0)$ or $(c_2,\dots,c_i+u_i-1,\dots,c_n+u_n) \neq (0)$. Hence $b_t = b_{t_o} \in k$. But this implies in turn that $\beta \in k$, contradicting the initial assumption that $\alpha A_n + \beta A_n \neq A_n$.

Thus, $t \geq 1$ and we complete the proof by computing b_{t-1} and b_{t-2}. From the coefficient of y_1^t in Equation 3.4 we get

$$0 = \sum_{j=2}^{n} \lambda_j [x_j y_j, b_{t-1}] - p b_{t-1} + \sum_{j=2}^{n} [x_j + y_j, b_t]. \tag{3.7}$$

We solve this for b_{t-1}. Recall that by Lemma 3.2.2 $\theta(s)$ has the same degree with s. First, as $\deg(\sum[x_i+y_i,b_t]) < \deg b_t$, by Lemma 3.2.2 there exists $f \in S$ with $\deg f < \deg b_t$, such that $0 = \theta(f) + \sum[x_j+y_j,b_t]$. This means that $\theta(b_{t-1}-f) = 0$ and so by Lemma 3.2.2 we have $b_{t-1} = f + g$ where

$$g = \sum g_{\gamma'} x_2^{c_2'} \dots x_n^{c_n'} y_2^{c_2'+u_2} \dots y_n^{c_n'+u_n}$$

for some $g_{\gamma'} \in k$.

Finally, we consider the coefficient of y_1^{t-1} in Equation 3.4 which is

$$0 = t b_t + \sum_{j=2}^{n} \lambda_j [x_j y_j, b_{t-2}] - p b_{t-2} + \sum_{j=2}^{n} [x_j + y_j, b_{t-1}] \tag{3.8}$$

where b_{t-2} is defined to be zero if $t = 1$. Suppose that $\deg b_t = \gamma = (c_2,\dots,c_n,c_2+u_2,\dots,c_n+u_n)$ and consider the coefficient of z^γ in Equation 3.8. By Lemma 3.2.2 again, $\sum_j [x_j y_j, b_{t-2}] - p b_{t-2} = \theta(b_{t-2})$ has no term of degree γ (because the terms of degree γ becomes zero by the property of θ). As can be seen from Equation 3.6, $\sum[x_j+y_j,g]$ also has no term in this degree, while $\deg\sum[x_j+y_j,f] < \deg f < \deg b_t$. In other words, the coefficient of z^γ in Equation 3.8 is just $0 = t b_{t-1}$. Since $t > 0$ this is impossible. Thus, αA_n indeed is a maximal right ideal of A_n. □

Let α be as above. Let $M = A_n / x_1 \alpha A_n$ and $S = A_n/\alpha A_n$ be the simple right A_n-module of Theorem 3.2.3.

Lemma 3.2.4 *M is an essential extension of the simple right A_n-module S which has Krull dimension $n-1$.*

Proof: There is an injective map $f : A_n/\alpha A_n \to A_n/x_1\alpha A_n$ given by $r + \alpha A_n \mapsto x_1 r + x_1\alpha A_n$ and we identify S with its image $x_1 A_n/x_1\alpha A_n$.

Let $R = A_{n-1}(k)$, with generators $x_2,\dots,x_n$ and $y_2,\dots,y_n$ subject to the relations

$$x_i y_j = y_j x_i + \delta_{ij} \quad \forall\ 2 \le i,j \le n.$$

Set $S = R[y]$ and $T = S[x; \frac{\partial}{\partial y}]$. Then $T \simeq A_n(k)$. By Theorem 3.2.3, the element

$$\alpha = x + yu + v \in T$$

generates a maximal right ideal where $u, v \in R$ are $u = \sum_{i=2}^{n} \lambda_i x_i y_i$ and $v = \sum_{i=2}^{n} x_i + y_i$, with $\lambda_2,\dots,\lambda_n \in k$ are linearly independent over $\mathbb{Q}$.

Each element in T can be written uniquely as a polynomial in x with coefficients in S. Hence we can talk about the x-degree of an element f of T, which we denote by $\deg_x(f)$.

Write $\alpha = x+f$ where $f = yu+v \in R[y]$. Note that $R[y] \cap \alpha T = 0$ and that any element in $T/\alpha T$ can be uniquely represented by a polynomial in $R[y]$: let $\gamma = \sum_{i=0}^{m} x^i g_i \in T$ with $g_i \in R[y]$. Then

$$\gamma + \alpha T = \sum_{i=0}^{m} (-f) x^{i-1} g_i + \alpha T$$

and by using $f x^{i-1} = x^{i-1} f - (i-1) x^{i-2} \partial_y(f)$, we can represent $\gamma + \alpha T$ through an element with lower x-degrees. Hence repeating these substitutions m times leads to a representation of $\gamma + \alpha T$ by a polynomial in $R[y]$. This means that the simple module $T/\alpha T$ of Theorem 3.2.3 has a basis consisting of the classes represented by a basis of $R[y]$. In particular, any element of $xT/x\alpha T$ can be represented as $xh + x\alpha T$ with $h \in R[y]$.

Sublemma 3.2.5 *Any element γ of $T/x\alpha T$ can be represented by $g + xh + x\alpha T$ where $g, h \in R[y]$. If γ is a nonzero element of $T/x\alpha T$, then γx is also nonzero. Concretely, if γ is represented by $g + xh$, then γx is represented by*

$$\gamma x = -\partial_y(g) + x(g + fh - \partial_y(h)) + x\alpha T.$$

Proof: By going to the factor $T/x\alpha T \to T/xT$ and using the fact that $(R[y]+xT)/xT = T/xT$, there exist $g \in R[y]$ and $r \in xT$ such that $\gamma = g+r+x\alpha T$. Write $r = xh$ with $h \in R[y]$ as noted in the paragraph preceding the sublemma. Note that

$$xhx = x^2h - x\partial_y(h) \equiv x(fh - \partial_y(h)) \pmod{x\alpha T}.$$

Then

$$\gamma x = xg - \partial_y(g) + x(fh - \partial_y(h)) + x\alpha T = -\partial_y(g) + x(fh - \partial_y(h) + g) + x\alpha T.$$

If γx were zero, then $\partial_y(g) = 0$ and $fh - \partial_y(h) + g = 0$. The first condition implies that $g \in R$ while the second condition leads to a contradiction since fh has higher y-degree than $-\partial_y(h) + g$ which means that $fh = 0$ as R is a domain. Thus $h = 0$ and also $g = 0$, contradicting that γ is nonzero. □

We now prove that any nonzero right submodule of $T/x\alpha T$ contains a nonzero element of $xT/x\alpha T$. By the above sublemma, any nonzero element $\gamma \in U$ of a nonzero submodule U of $T/x\alpha T$ is represented by $\gamma = g + xh$ with $g, h \in R[y]$. Multiplying by x on the right leads to an element $\gamma x = -\partial_y(g) + xh' + x\alpha T \in U$. Multiplying γ by x on the right $\deg_y(g)+1$ times leads to a nonzero element in $xT/x\alpha T$. Hence, M is an essential extension of S.

We now show that M has Krull dimension $n-1$. Note that $\mathrm{K.dim}(M) = \mathrm{K.dim}(M/S) = \mathrm{K.dim}(A_n/x_1A_n)$. Every element of A_n/x_1A_n can be written as a polynomial in y_1 with coefficients in A_{n-1}. Let U be an A_n-submodule of A_n/x_1A_n. Then the set $V = U \cap A_{n-1} = \{f \in A_{n-1} \mid f + x_1A_n \in U\}$ is a right ideal of A_{n-1} and $V[y_1] \subseteq U$. On the other hand, for any $p = \sum_{i=0}^m f_iy^i + x_1A_n$, we have $px^m = c_mf_m + x_1A_n \in U$ for some $c_m \in k$ and so $f_m \in V$. This implies that $f_my^m + x_1A_n \in U$ too, which gives in turn that $p' = \sum_{i=0}^{m-1} f_iy^i + x_1A_n \in U$. From this we get upon multiplying on the right by x^{m-1} that $f_{m-1} \in V$. Going on this way we get $f_i \in V$ for all $i = 0, \ldots, m$ and hence $U \subseteq V[y_1]$. Hence $U = V[y_1]$.

Consider the mapping $V \mapsto V[y_1]$ between the lattice of right ideals of A_{n-1} and the lattice of right A_n-submodules of A_n/x_1A_n. The above paragraph shows that this is onto.

Moreover, for two right ideals I and J of A_{n-1}, the equality $I[y_1] = J[y_1]$ implies that $I = J$. In the case $I \subseteq J$ it is easy to check that $I[y_1] \subseteq J[y_1]$ and this mapping is indeed a lattice isomorphism. Hence, the Krull dimension of A_n/x_1A_n as a right A_n-module is equal to the Krull dimension of A_{n-1} as a right module over itself, which is $n-1$. □

Hence we showed that Stafford's result holds in a more general setting and we record the following corollary.

Corollary 3.2.6 *Let k be a field which is at least $n-1$-dimensional over $\mathbb{Q}$. Then the Weyl algebra $A_n(k)$ has property $(\diamond)$ if and only if $n = 1$.*

3.3 Noetherian rings with enough normal elements

As indicated in the introduction, we examine the role played by the normal elements on property $(\diamond)$ for a Noetherian ring.

A module M is a **subdirect product** of a family of modules $\{F_\lambda\}_\Lambda$ if there exists an embedding $\iota : M \to \prod_\Lambda F_\lambda$ into a product of the modules F_λ such that for each projection $\pi_\mu : \prod F_\lambda \to F_\mu$ the composition $\pi_\mu \iota$ is surjective. Consequently, a module N is isomorphic to a subdirect product of the family $\{M_\lambda\}_\Lambda$ if and only if there is a family of epimorphisms $f_\lambda : N \to M_\lambda$ such that $\cap_\Lambda \ker f_\lambda = 0$. The following is a standart result in module theory.

Lemma 3.3.1 *Any nonzero module is isomorphic to a subdirect product of factor modules that are essential extensions of a simple module.*

Proof: For a proof of this fact see for instance [67, (14.9)]. □

The modules which are essential extensions of a simple module are known in the literature as **subdirectly irreducible**, **cocyclic**, **colocal**, or **monolithic**. If R is a ring with property $(\diamond)$ and M is a subdirectly irreducible left R-module with an essential simple module S, then $E(M) = E(S)$ is locally Artinian and hence M is locally Artinian as well. Conversely, if all the subdirectly irreducible R-modules are locally Artinian, then in particular the injective hulls of simple modules are locally Artinian and hence R has property $(\diamond)$. We just proved:

Lemma 3.3.2 *A ring R has property $(\diamond)$ if and only if subdirectly irreducible R-modules are locally Artinian.*

As we mentioned before, Hirano considered in [28] the stronger property that every left R-module M of finite length has an injective hull of finite length. In particular, he proved in [28, Theorem 1.1] that injective hulls of simple left R-modules have finite length if and only if every left R-module is a subdirect product of the family $\{N \leq M \mid M/N \text{ has finite length}\}$. This should be compared with our next result.

Lemma 3.3.3 *A ring R has property $(\diamond)$ if and only if every left R-module is a subdirect product of locally Artinian modules.*

Proof: Since, by the above lemma, property $(\diamond)$ is equivalent to subdirectly irreducible modules to be locally Artinian, the result follows by using Lemma 3.3.1. □

An element a of a ring R is called a **normal element** if $aR = Ra$. For instance, any central element is normal. A ring extension $R \subseteq S$ is said to be a **finite normalizing extension** if there exists a finite set $\{a_1, a_2, \ldots, a_k\}$ of elements of S such that $S = \sum_{i=1}^{k} a_i R$ and $a_i R = R a_i$.

Let $R \subseteq S$ be a finite normalizing extension with $\{a_1, a_2, \ldots, a_k\}$ being a set of elements of S which normalize R. Let M be a left S-module. M is also a left R-module by restriction. For an R-submodule N of M, let us denote by $a_i^{-1}N$ the set $\{m \in M \mid a_i m \in N\}$. Note that since a_i is normalizing, each $a_i^{-1}N$ is an R-submodule of M. Moreover, there is a largest S-submodule of M contained in N. This is called the **bound** of N and denoted by $b(N)$. In fact, $b(N) = \bigcap_{i=1}^{t} a_i^{-1}N$, see [47, 10.1.7].

Lemma 3.3.4 *[47, 10.1.6] With the notation of the preceding paragraph, the map $M/a_i^{-1}N \to M/N$ given by $m + a_i^{-1}N \mapsto a_i m + N$ is a group monomorphism. It induces a lattice embedding $\mathcal{L}(M/a_i^{-1}N)_R \to \mathcal{L}(M/N)_R$ under which finitely generated submodules are* *carried over to finitely generated submodules. If a_i centralizes R the map is an R-homomorphism.*

We mentioned that Hirano showed that the properties of being a co-Noetherian ring and having all injective hulls of simple modules finite length are carried to finite normalizing extensions. In this direction we prove the following result, which is actually an adaptation of Hirano's result [28, Theorem 1.8]:

Proposition 3.3.5 *Let S be a finite normalizing extension of a ring R. If R has property $(\diamond)$ then so does S.*

Proof: Let M be a nonzero left S-module. By Lemma 3.3.3, there exists a finite family $\{N_\lambda\}$ of R-submodules of M such that M/N_λ is locally Artinian for all λ and $\bigcap_\lambda N_\lambda = 0$. Since $b(N_\lambda) \subseteq N_\lambda$, we certainly have $\bigcap_\lambda b(N_\lambda) = 0$. By Lemma 3.3.4, there is a lattice embedding of R-modules $\mathcal{L}(M/b(N_\lambda)) \to \mathcal{L}(M/N_\lambda)$ which implies also that $M/b(N_\lambda)$ is locally Artinian. Hence M is a subdirect product of locally Artinian S-modules and the result follows from Lemma 3.3.3. □

What we have proved up to now enables us to prove the following, which says that tensoring with finite dimensional algebras preserves property $(\diamond)$.

Corollary 3.3.6 *Let C be a finite dimensional algebra over some field k and A be any algebra. If A has property $(\diamond)$ then $C \otimes A$ has property $(\diamond)$ too.*

Proof: Let $\{x_1, x_2, \ldots, x_n\}$ be a basis of C. Then we have $C \otimes A = \sum_{i=1}^n (x_i \otimes 1)A$ where each $x_i \otimes 1$ is a normal element and so $C \otimes A$ is a finite normalizing extension of A. Hence $C \otimes A$ has property $(\diamond)$ by Proposition 3.3.5. □

A sequence $x_1, \ldots, x_n$ of elements of a ring R is called a **normalizing** (resp. **centralizing**) **sequence** if for each $j = 0, \ldots, n-1$ the image of x_{j+1} in $R/\sum_{i=1}^j x_i R$ is a normal (resp. central) element. McConnell showed in [46] that every ideal in the enveloping algebra of a finite dimensional nilpotent Lie algebra has a centralizing sequence of generators. In the next section we will show a super version of his result.

Theorem 3.3.7 *[47, 4.2.2] The following conditions on an ideal A of a left Noetherian ring R are equivalent:*

(a) If $I \leq R$ is a left ideal of R, then $I \cap A^n \subseteq AI$ for some n.

(b) If ${}_R M$ is finitely generated and $N \leq M$ is a submodule of M, then $N \cap A^n M \subseteq AN$ for some n.

(c) If ${}_R M$ is finitely generated and $N \leq_e M$ with $AN = 0$ then $A^n M = 0$ for some n.

An ideal A of a left Noetherian ring is said to satisfy the left **Artin-Rees property** if it satisfies one of the equivalent conditions of the above theorem. If every ideal A of a ring R has the Artin-Rees property, then R is called an **Artin-Rees ring**.

Normal elements and ideals generated by such elements are important for our study because if R is a left Noetherian ring and A is an ideal of R generated by normal elements, then A has the left Artin-Rees property [47, 4.2.6].

The Artin-Rees property plays a central role in the following result, which is the first step towards the main result of this section.

Lemma 3.3.8 *Let A be a Noetherian algebra, E be a simple left A-module and $E \subseteq_e M$ be an essential extension of left A-modules. Let $Q \subseteq Ann_A(E)$ be an ideal of A that has a normalizing sequence of generators. Then M is Artinian if and only if $M' = Ann_M(Q)$ is Artinian.*

Proof: The proof is by induction on the number of elements of the generating set of Q. First suppose that $Q = \langle x_1 \rangle$ where x_1 is a normal element. Define a map $f : M \to M$ by $f(m) = x_1 m$. This map is $Z(A)$-linear and preserves A-submodules of M, because if $U \leq M$ is an A-submodule of M, then $A \cdot f(U) = Ax_1 U = x_1 AU = x_1 U = f(U)$ and so $f(U)$ is an A-submodule of M. Since Q is generated by a normal element, it satisfies the Artin-Rees property and so there exists a natural number $n > 0$ such that $Q^n M = x_1^n M = 0$. In other words, $\ker(f^n) = M$. Hence we have a finite filtration

$$0 \subseteq \ker(f) = \mathrm{Ann}_M(Q) \subseteq \ker(f^2) \subseteq \cdots \subseteq \ker(f^{n-1}) \subseteq \ker(f^n) = M$$

whose subfactors are left A/Q-modules and f induces a submodule preserving chain of embeddings

$$M/\ker(f^{n-1}) \hookrightarrow \ker(f^{n-1})/\ker(f^{n-2}) \hookrightarrow \cdots \hookrightarrow \ker(f^2)/\ker(f) \hookrightarrow \ker(f).$$

Hence M is Artinian if and only if $M' = \ker(f) = \mathrm{Ann}_M(Q)$ is Artinian.

Now let $n > 0$ and suppose that the assertion holds for all Noetherian algebras and finitely generated essential extensions $E \leq_e M$ of simple left A-modules E such that $\mathrm{Ann}_A(E)$ contains an ideal Q which has a normalizing sequence of generators with less than n elements. Let $E \leq_e M$ be a finitely generated essential extension of a simple left A-module such that $Q \subseteq \mathrm{Ann}_A(E)$ has a normalizing sequence of generators $\{x_1, \ldots, x_n\}$ of n elements. Consider the submodule $M' = \mathrm{Ann}_M(x_1)$. Since x_1 is a normal element, we can apply the same procedure to conclude that M is Artinian if and only if M' is Artinian. Let $A' = A/Ax_1$ and $Q' = Q/Ax_1$. Then $Q' \subseteq \mathrm{Ann}_{A'}(E)$ is generated by the set $\{\overline{x_2}, \ldots, \overline{x_n}\}$ of normalizing elements, where $\overline{x_i}$ is the image of x_i in A' for $i = 2, \ldots, n$. Now, $E \leq M'$ is an essential extension of A'-modules such that $Q'E = 0$. Since Q' is generated by a normalizing sequence of $n-1$ elements, by the induction hypotheses we conclude that M is Artinian if and only if $\mathrm{Ann}_{M'}(Q)' = \mathrm{Ann}_M(Q)$ is Artinian as A'-modules and hence also as A-modules. □

Lemma 3.3.9 *Suppose that A is a Noetherian algebra such that every primitive ideal P of A contains an ideal $Q \subseteq P$ which has a normalizing sequence of generators and A/Q has property $(\diamond)$. Then A has property $(\diamond)$.*

Proof: Let E be a simple left A-module, $P = \mathrm{Ann}_A(E)$ and let $E \subseteq_e M$ be a finitely generated essential extension of E. Let $M' = \mathrm{Ann}_M(Q)$, where $Q \subseteq P$ is an ideal that has a normalizing sequence of generators and with A/Q having property $(\diamond)$. Then $E \leq M'$ is a finitely generated essential extension of A/Q-modules and so M' is Artinian because A/Q has property $(\diamond)$. Since by Lemma 3.3.8 M' is Artinian if and only if M is Artinian, it follows that M is Artinian and A has property $(\diamond)$. □

Let $A = A_0 \oplus A_1$ be an associative superalgebra. A **graded primitive ideal** P of A is the annihilator of a graded simple A-module, while a **graded maximal ideal** is a proper graded ideal that is a maximal element in the lattice of proper graded ideals. Given any ideal P of A, the set $Q = P \cap \sigma(P)$ is a graded ideal where σ denotes the involution

$$\sigma : A \to A, \quad a_0 + a_1 \mapsto a_0 - a_1 \quad \forall a_0 \in A_0, a_1 \in A_1.$$

We remark that if A is a superalgebra over a field k of characteristic zero and I is a graded ideal of A, then Bell and Musson prove that I is graded maximal if and only if $I = \sigma(P) \cap P$ for a maximal ideal P of A [3, Lemma 1.2]. This fact will be used in the proof of the main result of this section.

We are ready to prove the main result of this section. It states that for certain associative Noetherian superalgebras, to decide whether they have property $(\diamond)$ it is enough to look at the primitive factors. This way we obtain a reduction of the problem to the primitive factors.

Theorem 3.3.10 *Let A be a Noetherian associative superalgebra over a field k of characteristic zero such that every primitive ideal is maximal and every graded maximal ideal is generated by a normalizing sequence of generators. Then the following statements are equivalent:*

(a) A has property $(\diamond)$.

(b) Every primitive factor of A has property $(\diamond)$.

(c) Every graded primitive factor of A has property $(\diamond)$.

Proof: The part $(a) \Rightarrow (b)$ is clear since property $(\diamond)$ is inherited by factor rings.

$(b) \Rightarrow (c)$ Suppose that Q is a graded primitive ideal of A. By Bell and Musson's result, there exists a maximal ideal P of A such that $Q = P \cap \sigma(P)$. If Q is graded, then $Q = P$ and A/Q has property $(\diamond)$ by hypothesis. Otherwise, by the maximality of P, $\sigma(P) + P = A$ holds. In this case the map $A/Q \to A/P \times A/\sigma(P)$ given by $a + Q \mapsto (a + P,\ a + \sigma(P))$ for $a \in A$ is an isomorphism and so $A/Q \simeq A/P \times A/\sigma(P)$. Since A/P has property $(\diamond)$, so does $A/\sigma(P)$ and then also the direct product of both has property $(\diamond)$.

$(c) \Rightarrow (a)$ Suppose that every graded primitive factor of A has property $(\diamond)$. Let E be a simple A-module, $P = \mathrm{Ann}_A(E)$, and let $E \leq M$ be a finitely generated essential extension of E. P is maximal by assumption. The ideal $Q = P \cap \sigma(P)$ is graded maximal by Bell and Musson's result and has a normalizing sequence of generators by assumption. A/Q has property $(\diamond)$ by the hypothesis and by Lemma 3.3.9 we conclude that A has property $(\diamond)$. □

3.4 Ideals in enveloping algebras of nilpotent Lie superalgebras

In the previous section we obtained a possible reduction of the problem to the study of primitive factors under certain assumptions for Noetherian associative superalgebras. In this section we prove that the universal enveloping algebra of a finite dimensional nilpotent Lie superalgebra satisfies these assumptions. We will start with the previously announced analogue of McConnell's result which says that every ideal of the enveloping algebra of a finite dimensional nilpotent Lie algebra has a centralizing sequence of generators. To arrive at the conclusions of this section we will study some aspects of locally nilpotent derivations of superalgebras.

Let A be an associative superalgebra. We define the **supercommutator** of two homogeneous elements a, b of A as the element

$$[\![a,b]\!] \ := \ ab - (-1)^{|a||b|}ba$$

which is extended bilinearly to a form $[\![-,-]\!] : A^{\otimes 2} \to A$. The **supercenter** of A is the set $Z(A)_s = \{a \in A \mid \forall \ b \in A : [\![a,b]\!] = 0\}$ and its elements are called **supercentral**. Supercentral elements are clearly normal. A **superderivation** of a superalgebra A is a graded linear map $f : A \to A$ of degree $|f|$ such that

$$f(ab) = f(a)b + (-1)^{|a||f|}af(b)$$

for all homogeneous $a, b \in A$. The supercommutator $[\![x,-]\!]$ for a homogeneous element $x \in A$ is an example of a superderivation. Such derivations are called **inner derivation**. If $|a| = 0$, then $[\![a,-]\!]$ is a derivation of A.

Proposition 3.4.1 *Let A be a superalgebra and f be a superderivation of A. For every $n \in \mathbb{N}$ and homogeneous elements a, b of A, there exist integers $c_0, \ldots, c_n$ such that $f^n(ab) = \sum_{i=0}^n c_i f^i(a) f^{n-i}(b)$.*

Proof: Let a and b be homogeneous elements of A. We use induction on n. The case $n = 1$ follows from the definition of a superderivation with $c_0 = (-1)^{|a||b|}$ and $c_1 = 1$. Suppose that the assertion holds for $n \geq 1$. We compute $f^{n+1}(ab)$:

$$\begin{aligned}
f^{n+1}(ab) &= f\left(\sum_{i=0}^n c_i f^i(a) f^{n-i}(b)\right) = \sum_{i=0}^n c_i (f^{i+1}(a) f^{n-i}(b) + (-1)^{|f^i(a)||f|} f^i(a) f^{n-i+1}(b)) \\
&= \sum_{i=1}^{n+1} c_{i-1} f^i(a) f^{n+1-i}(b) + \sum_{i=0}^n (-1)^{|f^i(a)||f|} c_i f^i(a) f^{n-i+1}(b) \\
&= (-1)^{|a||f|} c_0 a f^{n+1}(b) + \sum_{i=1}^n ((c_{i-1} + (-1)^{|f^i(a)||f|} c_i) f^i(a) f^{n+1-i}(b)) + c_n f^{n+1}(a) b \\
&= \sum_{i=0}^{n+1} c'_i f^i(a) f^{n+1-i}(b)
\end{aligned}$$

where $c'_0 = (-1)^{|a||f|} c_0$, $c'_{n+1} = c_n$ and $c'_i = c_{i-1} + (-1)^{|f^i(a)||f|} c_i$ for all $i = 1, \ldots, n$. □

Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a Lie superalgebra and choose a basis $\{x_1, \ldots, x_n\}$ of $\mathfrak{g}_0$ and a basis $\{y_1, \ldots, y_m\}$ of $\mathfrak{g}_1$, and let $A = U(\mathfrak{g})$ be the enveloping algebra of $\mathfrak{g}$. By the PBW theorem for Lie superalgebras, the monomials $x_1^{\alpha_1} \ldots x_n^{\alpha_n} y_1^{\beta_1} \ldots y_m^{\beta_m}$ with α_i, $\beta_j \in \mathbb{N}_0$ and $\beta_i \leq 1$ form a basis for the enveloping algebra A [4, Theorem 1].

If we let

$$A_i = \operatorname{span}\{x_1^{\alpha_1} \cdots x_n^{\alpha_n} y_1^{\beta_1} \cdots y_m^{\beta_m} \mid \beta_1 + \cdots + \beta_m = i \pmod 2\}$$

for $i = 0, 1$, then $A = A_0 \oplus A_1$ is an associative superalgebra such that the degree of a homogeneous element of $\mathfrak{g}$ equals its degree in A.

The adjoint action of an element x of $\mathfrak{g}$ on A is defined by

$$\mathrm{ad}_x : A \to A, \quad \mathrm{ad}_x(a) = [\![x, a]\!] \quad \forall a \in A.$$

By the definition of the enveloping algebra, we have for all $x, y \in \mathfrak{g}$:

$$\mathrm{ad}_x(y) = [\![x, y]\!] = [x, y].$$

A map $f : A \to A$ is called **locally nilpotent** if for every $a \in A$ there exists a number $n(a) \geq 0$ such that $f^{n(a)}(a) = 0$. Recall that we defined in § 1.4.1.1 the nilpotency degree of a nilpotent Lie algebra to be the least positive integer r such that $\mathfrak{g}^r = 0$. We will show that if $\mathfrak{g}$ is a finite dimensional nilpotent Lie superalgebra, then the adjoint action of each homogeneous element $x \in \mathfrak{g}$ is a locally nilpotent superderivation. In this direction we first prove the following result which follows from a direct computation.

Lemma 3.4.2 *For any $x, y \in \mathfrak{g}$ one has*

$$\mathrm{ad}_x \circ \mathrm{ad}_y - (-1)^{|x||y|} \mathrm{ad}_y \circ \mathrm{ad}_x = \mathrm{ad}_{[x,y]}. \tag{3.9}$$

Proof: Let a be a homogeneous element of A and let $x, y \in \mathfrak{g}$.

$$\begin{aligned}
[\![x,[\![y,a]\!]]\!] - (-1)^{|x||y|}[\![y,[\![x,a]\!]]\!] &= x(ya-(-1)^{|y||a|}ay) - (-1)^{|x|(|y|+|a|)}(ya-(-1)^{|y||a|}ay)x \\
&\quad - (-1)^{|x||y|}\Big[y(xa-(-1)^{|x||a|}ax) - (-1)^{|y|(|x|+|a|)}(xa-(-1)^{|x||a|}ax)y\Big] \\
&= xya + (-1)^{|x||y|+|x||a|+|y||a|}ayx - (-1)^{|x||y|}yxa - (-1)^{|a||y|+|x||a|}axy \\
&= [x,y]a + (-1)^{|a|(|x|+|y|)}a[x,y] = [\![[x,y],a]\!].
\end{aligned}$$

□

Proposition 3.4.3 *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra. Then ad_x is a locally nilpotent superderivation of $A = U(\mathfrak{g})$, for every homogeneous element $x \in \mathfrak{g}$.*

Proof: Let r be the nilpotency degree of $\mathfrak{g}$, *i.e.* $\mathfrak{g}^r = 0$. Then for any $a \in \mathfrak{g}$ we have $\mathrm{ad}_x^r(a) = 0$. We proceed by induction on the length of the monomials. For monomials of length 1 the result is already true since $\mathfrak{g}$ is nilpotent. Let $m \geq 0$. Suppose that for every monomial $a \in A$ of length m there exists $n(a) \geq 0$ such that $\mathrm{ad}_x^{n(a)}(a) = 0$. Let $y \in \mathfrak{g}$. Then there exist integers $c_0, c_1, \ldots, c_{n(a)+r}$ such that

$$\mathrm{ad}_x^{n(a)+r}(ay) = \sum_{i=0}^{n(a)+r} c_i \, \mathrm{ad}_x^i(a) \, \mathrm{ad}_x^{n(a)+r-i}(y) = 0.$$

By induction ad_x is locally nilpotent on all basis elements of A. □

Given an l-tuple of superderivations $\partial = (\partial_1, \ldots, \partial_l)$ of a superalgebra A we say that a subset X of A is **∂-stable** if $\partial_i(X) \subseteq X$ for all $1 \leq i \leq l$. Note that if all superderivations ∂_i are inner, then any ideal I is ∂-stable. Given a homogeneous supercentral element $a \in A$, the ideal $I = Aa$ is graded and A/Aa is again a superalgebra, with the grading given by $\overline{A_i} = (A_i + I)/I$ for $i = 0, 1$. We say that a sequence $\{x_1, \ldots, x_n\}$ of homogeneous elements of a superalgebra is a **supercentralizing sequence** if for each $j = 0, \ldots, n-1$ the image of x_{j+1} in $A/\sum_{i=1}^{j} x_i A$ is a supercentral element.

Theorem 3.4.4 *Let k be a field of characteristic zero and A be a superalgebra over k with locally nilpotent superderivations $\partial_1, \ldots, \partial_l$ such that $\bigcap_{i=1}^{l} \ker \partial_i \subseteq Z(A)_s$ and for all $i \leq j$ there exist $\lambda_{i,j} \in k$ with*

$$\partial_i \circ \partial_j - \lambda_{i,j} \partial_j \circ \partial_i \in \sum_{s=1}^{i-1} k \partial_s. \tag{3.10}$$

Then any nonzero ∂-stable ideal I of A contains a nonzero supercentral element. In particular if I is graded and Noetherian, then it contains a supercentralizing sequence of generators consisting of homogeneous elements.

Proof: For each $1 \leq t \leq l$ set $K_t = \bigcap_{i=1}^{t} \ker \partial_i$. We will first show that K_i are ∂-stable subalgebras of A. Let $1 \leq t, j \leq l$ and $a \in K_t$. If $j \leq t$, then $\partial_j(a) = 0 \in K_t$ by definition. Hence suppose $j > t$. By hypothesis for any $1 \leq i \leq t < j$ we have

$$\partial_i(\partial_j(a)) = \lambda_{i,j} \partial_j(\partial_i(a)) + \sum_{s=1}^{i-1} \mu_{i,j,s} \partial_s(a) = 0$$

for some $\lambda_{i,j}, \mu_{i,j,s} \in k$. Thus $\partial_j(a) \in K_t$.

Let I be a ∂-stable ideal of A. We show that I contains a nonzero element of the supercenter of A. Note that since ∂_1 is locally nilpotent, for any $0 \neq a \in I$ there exists $n_1 \geq 0$ such that $0 \neq a' = \partial_1^{n_1}(a) \in \ker \partial_1 = K_1$. Since I is ∂_1-stable, $a' \in I \cap K_1$. Suppose $1 \leq t \leq l$ and $0 \neq a_t \in I \cap K_t$, then since ∂_{t+1} is locally nilpotent, there exists $n_{t+1} \geq 0$ such that $0 \neq a' = \partial_{t+1}^{n_{t+1}}(a) \in \ker \partial_{t+1}$. Since I and K_t are ∂-stable, we have $a' \in I \cap K_{t+1}$. Hence for $t = l$, we get $0 \neq I \cap K_l \subseteq I \cap Z(A)_s$.

Assume that I is graded and Noetherian and let $0 \neq a = a_0 + a_1 \in I \cap Z(A)_s$. Since I and $Z(A)_s$ are graded and a graded ideal contains the homogeneous components of all of its elements, both parts a_0 and a_1 belong to $I \cap Z(A)_s$, one of them being nonzero. Thus we might choose a to be homogeneous. Let $J_1 = Aa$ be the graded ideal generated by a. Since J_1 is ∂-stable, all superderivations ∂_i lift to superderivations of A/J_1 satisfying the same relation (3.10) as before. Moreover I/J_1 is a graded Noetherian ∂-stable ideal of A/J_1. Applying the procedure of obtaining a supercentral element to I/J_1 in A/J_1 yields a supercentral homogeneous element $a' + J_1 \in I/J_1 \cap Z(A/J_1)_s$. Set $J_2 = Aa + Aa'$. Continuing in this way leads to an ascending chain of ideals $J_1 \subseteq J_2 \subseteq \cdots \subseteq I$ that eventually has to stop, *i.e.* $I = J_m$ for some m. By construction, the generators used to build up $J_1, J_2, \ldots, J_m$ form a supercentralizing sequence of generators for I. □

We now show that the enveloping algebra of a finite dimensional nilpotent Lie superalgebra has a set of nilpotent superderivations which satisfy the assumptions of the previous theorem, and conclude that any graded ideal of such an enveloping algebra has a supercentralizing sequence of generators. In order to do so we choose an appropriate basis of homogeneous elements. If $\mathfrak{g}$ is a finite dimensional nilpotent Lie superalgebra, then $\mathfrak{g}$ has a refined central series

$$\mathfrak{g} = \mathfrak{g}(n) \supset \mathfrak{g}(n-1) \supset \mathfrak{g}(n-2) \supset \cdots \supset \mathfrak{g}(1) \supset \mathfrak{g}(0) = \{0\},$$

with $[\mathfrak{g}, \mathfrak{g}(i)] \subseteq \mathfrak{g}(i-1)$ and $\dim(\mathfrak{g}(i)/\mathfrak{g}(i-1)) = 1$ for all $1 \leq i \leq n$. Let $x_1, x_2, \ldots, x_n$ be a basis of $\mathfrak{g}$ such that each element $x_i + \mathfrak{g}(i-1)$ is nonzero (and hence forms a basis) in $\mathfrak{g}(i)/\mathfrak{g}(i-1)$. Actually each x_i is homogeneous, since if $x_i = x_{i0} + x_{i1}$ with x_{ij} homogeneous, then as x_{i0} and x_{i1} cannot be linearly independent as $\mathfrak{g}(i)/\mathfrak{g}(i-1)$ is 1-dimensional, one of them belongs to $\mathfrak{g}(i-1)$.

Corollary 3.4.5 *Any graded ideal of the enveloping algebra of a finite dimensional nilpotent Lie superalgebra has a supercentralizing sequence of generators consisting of homogeneous elements.*

Proof: Let $\mathfrak{g}$ and $A = U(\mathfrak{g})$ be as above, as well as the chosen basis $x_1, \ldots, x_n$ of $\mathfrak{g}$ of homogeneous elements. Set $\partial_i = \mathrm{ad}_{x_i}$. By Proposition 3.4.3 all superderivations ∂_i are locally nilpotent. Let $i < j$, then $[x_i, x_j] \in \mathfrak{g}(i-1)$ shows that there are scalars $\mu_{i,j,s} \in k$ such that

$$[x_i, x_j] = \sum_{s=1}^{i-1} \mu_{i,j,s} x_s$$

Note that $\mathrm{ad}_{[x_i,x_j]} = \sum_{s=1}^{i-1} \mu_{i,j,s}\, \mathrm{ad}_{x_s}$. Therefore, using Lemma 3.4.2, we have

$$\partial_i \circ \partial_j = (-1)^{|x_i||x_j|} \partial_j \circ \partial_i + \sum_{s=1}^{i-1} \mu_{i,j,s} \partial_s.$$

Hence the assumptions of Theorem 3.4.4 are fulfilled and our claim follows since A is Noetherian. □

If $U(\mathfrak{g})$ is the enveloping algebra of a finite dimensional nilpotent Lie algebra, then every primitive ideal of $U(\mathfrak{g})$ is maximal by [14, 4.7.4]. In [43, 1.6], Letzter proves that this is carried over to finite extensions of $U(\mathfrak{g})$. Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional nilpotent Lie superalgebra. Then $\mathfrak{g}_0$ is a nilpotent Lie algebra. Since $U(\mathfrak{g})$ is a finite extension of $U(\mathfrak{g}_0)$ [4, Proposition 2], it follows that every primitive ideal of the enveloping algebra of a finite dimensional nilpotent Lie superalgebra is maximal.

With these remarks and Theorem 3.3.10 we are ready to prove the last result of this section.

Corollary 3.4.6 *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra. Then $U = U(\mathfrak{g})$ has property $(\diamond)$ if and only if every primitive factor of U does if and only if every graded primitive factor of U does.*

Proof: By Corollary 3.4.5 any graded ideal is generated by supercentral hence normal elements. Moreover every primitive ideal of $U(\mathfrak{g})$ is maximal by the preceding remarks. Hence the result follows from Theorem 3.3.10. □

3.5 Primitive factors of nilpotent Lie superalgebras

Now that we have reduced the problem to the primitive factors, in this section we will study such factors of nilpotent Lie superalgebras. The primitive factors of the enveloping algebra of a finite dimensional nilpotent Lie algebra are known to be Weyl algebras (see for example [14, Chapter 6]).

Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra over an algebraically closed field k of characteristic zero. Bell and Musson showed in [3] that the graded primitive factors of the enveloping algebra of a finite dimensional nilpotent Lie superalgebra are of the form $\mathrm{Cliff}_q(k) \otimes \mathrm{A}_p(k)$ where $\mathrm{Cliff}_q(k)$ is the Clifford algebra, which is defined as

$$\mathrm{Cliff}_0(k) = k, \quad \mathrm{Cliff}_1(k) = k \times k, \quad \mathrm{Cliff}_2(k) = \mathrm{M}_2(k)$$

and $\mathrm{Cliff}_{n+2}(k) = \mathrm{Cliff}_n(k) \otimes \mathrm{M}_2(k)$ for all $n > 2$. We have already seen in Corollary 3.3.6 that property $(\diamond)$ is preserved under tensoring by a finite dimensional algebra. The next result shows that the converse also holds if the finite dimensional algebra in question is a Clifford algebra.

Lemma 3.5.1 *Let k be a field. A $\mathbb{C}$-algebra A has property $(\diamond)$ if and only if $\mathrm{Cliff}_q(k) \otimes A$ has property $(\diamond)$ for all (for one) q.*

Proof: Since Clifford algebras are finite dimensional, by Corollary 3.3.6, $\mathrm{Cliff}_q(k) \otimes A$ has property $(\diamond)$ if A does. On the other hand suppose that there exists $q > 0$ such that $\mathrm{Cliff}_q(k) \otimes A$ has property $(\diamond)$. If $q = 2m$ is even, then $\mathrm{Cliff}_q(k) \otimes A = M_{2^m}(A)$, which is Morita equivalent to A. Since $(\diamond)$ is a Morita-invariant property as the equivalence between module categories yields lattice isomorphisms of the lattice of submodules of modules, we get that A has property $(\diamond)$. If $q = 2m+1$ is odd, then $\mathrm{Cliff}_q(k) \otimes A = M_{2^m}(A) \times M_{2^m}(A)$. Since A is Morita equivalent to the factor $M_{2^m}(A)$ it also has property $(\diamond)$. □

Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra and let $U(\mathfrak{g})$ be its enveloping algebra. We know that every primitive factor of $U(\mathfrak{g})$ is a tensor product of a Clifford algebra and a Weyl algebra and that property ($\diamond$) for such a tensor product depends on the Weyl algebra. In § 3.2 we have seen that the only Weyl algebra which has property ($\diamond$) is the first Weyl algebra. This suggests that one should study the primitive factors of $U(\mathfrak{g})$ to see when only the Weyl algebras of order less than or equal to one appear in such factors. Although the primitive factors of $U(\mathfrak{g})$ have been determined by Bell and Musson in [3], the order of the Weyl algebra appearing in such factors has been determined by Herscovich in [26] and is related to the so-called index of the underlying even part of the Lie superalgebra $\mathfrak{g}$.

Let $f \in \mathfrak{g}^*$ be a linear functional on a Lie algebra $\mathfrak{g}$ and set

$$\mathfrak{g}^f = \{x \in \mathfrak{g} \mid f([x,y]) = 0, \forall y \in \mathfrak{g}\}$$

be the orthogonal subspace of $\mathfrak{g}$ with respect to the bilinear form $f([-,-])$. The number

$$\mathrm{ind}(\mathfrak{g}) := \inf_{f \in \mathfrak{g}^*} \dim \mathfrak{g}^f$$

is called the **index** of $\mathfrak{g}$. Note that any functional $f \in \mathfrak{g}$ defines a symplectic form on the space $\mathfrak{g}/\mathfrak{g}^f$, and so the space $\mathfrak{g}/\mathfrak{g}^f$ has even dimension. Thus the index of a finite dimensional Lie algebra $\mathfrak{g}$ is of the form $\dim \mathfrak{g} - 2n$ for some $n \in \mathbb{N}$.

The following result relates the order of Weyl algebras appearing in the primitive factors of $U(\mathfrak{g})$ with the index of the even part of $\mathfrak{g}$.

Theorem 3.5.2 (Proposition 16 [26], Theorem A [3]) *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie superalgebra over an algebraically closed field k of characteristic zero. Then the following hold.*

(a) For $f \in \mathfrak{g}_0^*$ there exists a graded primitive ideal $I(f)$ of $U(\mathfrak{g})$ such that

$$U(\mathfrak{g})/I(f) \simeq \mathrm{Cliff}_q(k) \otimes A_p(k),$$

where $2p = \dim(\mathfrak{g}_0/\mathfrak{g}_0^f) \leq \dim(\mathfrak{g}_0) - ind\mathfrak{g}_0$ and $q \geq 0$.

(b) For every graded primitive ideal P of $U(\mathfrak{g})$ there exists $f \in \mathfrak{g}_0^*$ such that $P = I(f)$.

We now combine the above result with Corollary 3.4.6 and Stafford's result to obtain the following.

Proposition 3.5.3 *Let $\mathfrak{g} = \mathfrak{g}_0 \oplus \mathfrak{g}_1$ be a finite dimensional nilpotent Lie superalgebra over an algebraically closed field k of characteristic zero. Then $U(\mathfrak{g})$ has property $(\diamond)$ if and only if $ind(\mathfrak{g}_0) \geq \dim(\mathfrak{g}_0) - 2$.*

Proof: $(\Rightarrow)$ By Theorem 3.5.2 each graded primitive factor of $U(\mathfrak{g})$ is of the form $\mathrm{Cliff}_q(k) \otimes A_p(k)$ where $2p = \dim(\mathfrak{g}_0/\mathfrak{g}_0^f) = \dim(\mathfrak{g}_0) - \dim \mathfrak{g}_0^f$. Since property $(\diamond)$ is inherited by factor rings this implies together with Theorem 2.3.1 and Lemma 3.5.1 that $p \leq 1$, that is $\dim \mathfrak{g}_0^f \geq \dim(\mathfrak{g}_0) - 2$, *i.e.* $\mathrm{ind}\mathfrak{g}_0 \geq \dim(\mathfrak{g}_0) - 2$.
$(\Leftarrow)$ If $\mathrm{ind}\mathfrak{g}_0 \geq \dim(\mathfrak{g}_0) - 2$ then the graded primitive factors of $U(\mathfrak{g})$ are either of the form $\mathrm{Cliff}_q(k)$ or $\mathrm{Cliff}_q(k) \otimes A_1(k)$. Thus the graded primitive factors of $U(\mathfrak{g})$ have property $(\diamond)$ by Lemma 3.5.1. This implies together with Corollary 3.4.6 that $U(\mathfrak{g})$ has property $(\diamond)$. $\square$

3.6 Nilpotent Lie algebras with almost maximal index

In the previous section we saw that property $(\diamond)$ for a finite dimensional Lie superalgebra is controlled by the index of its even part. In this last section we will classify all finite dimensional nilpotent Lie algebras $\mathfrak{g}$ whose index is greater than or equal to $\dim \mathfrak{g} - 2$ and give the proof of the Main Theorem of this chapter. It is clear that if $\mathrm{ind}(\mathfrak{g}) = \dim \mathfrak{g}$, then all the brackets in $\mathfrak{g}$ are zero and hence $\mathfrak{g}$ is abelian. We say that a Lie algebra $\mathfrak{g}$ has **almost maximal index** if $\mathrm{ind}(\mathfrak{g}) = \dim \mathfrak{g} - 2$.

As a first step we show that a direct product $\mathfrak{g}_1 \times \mathfrak{g}_2$ of two Lie algebras $\mathfrak{g}_1$ and $\mathfrak{g}_2$ has almost maximal index if and only if one of them is abelian and the other one has almost maximal index. Recall that the Lie bracket of the direct product $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$ is defined as

$$[(x_1, y_1), (x_2, y_2)] := ([x_1, x_2], [y_1, y_2])$$

for all $x_1, x_2 \in \mathfrak{g}_1,\ y_1, y_2 \in \mathfrak{g}_2$. For the product algebra, we have the following formula.

Lemma 3.6.1 *For finite dimensional Lie algebras $\mathfrak{g}_1, \mathfrak{g}_2$ the following formula holds:*

$$ind(\mathfrak{g}_1 \times \mathfrak{g}_2) = ind(\mathfrak{g}_1) + ind(\mathfrak{g}_2).$$

In particular $\mathfrak{g}_1 \times \mathfrak{g}_2$ has almost maximal index if and only if one of the factors has almost maximal index and the other factor is Abelian.

Proof: Set $\mathfrak{g} = \mathfrak{g}_1 \times \mathfrak{g}_2$. Since $\mathfrak{g}^* = \mathfrak{g}_1^* \times \mathfrak{g}_2^*$, for all $f \in \mathfrak{g}^*$, we have $\dim \mathfrak{g}^f = \dim \mathfrak{g}_1^{f_1} + \dim \mathfrak{g}_2^{f_2}$, with $f_i = f\epsilon_i \in \mathfrak{g}_i^*$ and inclusions $\epsilon_i : \mathfrak{g}_i \to \mathfrak{g}$. Thus $\mathrm{ind}(\mathfrak{g}) = \mathrm{ind}(\mathfrak{g}_1) + \mathrm{ind}(\mathfrak{g}_2)$. Recall that in general $\mathrm{ind}(\mathfrak{g}_i) = \dim(\mathfrak{g}_i) - 2n_i$ for some $n_i \geq 0$. Hence

$$\mathrm{ind}(\mathfrak{g}) = \mathrm{ind}(g_1) + \mathrm{ind}(g_2) = \dim(\mathfrak{g}_1) - 2n_1 + \dim(\mathfrak{g}_2) - 2n_2 = \dim(\mathfrak{g}) - 2(n_1 + n_2) = \dim(\mathfrak{g}) - 2$$

if and only if $n_1 + n_2 = 1$ which shows our claim. □

In general it is unknown whether property ($\diamond$) is preserved under the formation of polynomial rings. However, in the case of the enveloping algebra of a finite dimensional nilpotent Lie algebra we have a positive result.

Proposition 3.6.2 *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra over an algebraically closed field k of characteristic zero. Then $U(\mathfrak{g})[x_1, \ldots, x_n]$ has property ($\diamond$) if and only if $U(\mathfrak{g})$ has property ($\diamond$).*

Proof: Suppose that $U(\mathfrak{g})$ has property ($\diamond$). We have

$$U(\mathfrak{g})[x_1, \ldots, x_n] = U(\mathfrak{g}) \otimes k[x_1, \ldots, x_n] = U(\mathfrak{g}) \otimes U(\mathfrak{a}) = U(\mathfrak{g} \oplus \mathfrak{a})$$

for an n-dimensional abelian Lie algebra $\mathfrak{a}$. Since $U(\mathfrak{g})$ has property ($\diamond$), $\mathfrak{g}$ has index at least $\dim(\mathfrak{g}) - 2$. By Lemma 3.6.1, we have $\mathrm{ind}(\mathfrak{g} \oplus \mathfrak{a}) \geq \dim(\mathfrak{g}) + n - 2 = \dim(\mathfrak{g} \oplus \mathfrak{a}) - 2$. Since $\mathfrak{g} \oplus \mathfrak{a}$ is nilpotent, it follows from Proposition 3.5.3 that $U(\mathfrak{g} \oplus \mathfrak{a})$ has property ($\diamond$). Thus $U(\mathfrak{g})[x_1, \ldots, x_n]$ also has property ($\diamond$). Conversely, if the polynomial algebra $U(\mathfrak{g})[x_1, \ldots, x_n]$ has property ($\diamond$), then so does $U(\mathfrak{g})$ since it is inherited by factor rings. □

By Lemma 3.6.1, we can ignore the abelian direct factors in the characterization of Lie algebras with almost maximal index. An element $f \in \mathfrak{g}^*$ is called **regular** if $\dim(\mathfrak{g}^f) =$

$\mathrm{ind}(\mathfrak{g})$. If f is a regular element of $\mathfrak{g}^*$, then the Lie algebra $\mathfrak{g}^f$ is abelian [14, Proposition 1.11.7]. Let k be a field of characteristic zero and let $\mathfrak{g}$ be a nilpotent Lie algebra over k of dimension n. Also assume that the maximal dimension of an abelian subalgebra is $n-2$. Burde and Ceballos show in [5, Proposition 5.1] that in this case there exists an algorithm to construct an abelian ideal of dimension $n-2$ from an abelian subalgebra of dimension $n-2$. This will be used in the proof of the following proposition.

Proposition 3.6.3 *Let $\mathfrak{g}$ be a finite dimensional nilpotent Lie algebra over a field k of characteristic zero. Then $\mathfrak{g}$ has almost maximal index if and only if $\mathfrak{g}$ has an abelian ideal of codimension 1 or if $\mathfrak{g}$ is isomorphic (up to an abelian direct factor) to $\mathfrak{h}_5$ or $\mathfrak{h}_6$.*

Proof: Suppose that $\mathfrak{g}$ does not have an abelian ideal of codimension one. Then there exists a linear function $f \in \mathfrak{g}^*$ such that $\dim(\mathfrak{g}^f) = n-2$. Then $\mathfrak{g}^f$ is an abelian Lie subalgebra of $\mathfrak{g}$. As we mentioned above, there exists in this case an abelian ideal $\mathfrak{a}$ of $\mathfrak{g}$ of codimension 2. Let $\{e_1, \dots, e_n\}$ be a basis of $\mathfrak{g}$ such that $\{e_3, \dots, e_n\}$ is a basis of $\mathfrak{a}$. Since $\mathfrak{a}$ is abelian, the matrix of brackets $[e_i, e_j]$ has the form

$$M = \Big([e_i, e_j]\Big) = \begin{pmatrix} A & B \\ -B^t & 0 \end{pmatrix}$$

where A is 2×2 skew-symmetric matrix and B is a $2\times(n-2)$ matrix with entries in $\mathfrak{a}$, and 0 is the $(n-2)\times(n-2)$ zero matrix. Since $\mathfrak{g}$ is nilpotent, $[e_1, e_2] \in \mathfrak{a}$. Moreover B cannot be the zero matrix since otherwise $\mathfrak{g}$ has an abelian ideal of codimension one. Let

$$M_{ij} = \begin{pmatrix} [e_1, e_i] & [e_1, e_j] \\ [e_2, e_i] & [e_2, e_j] \end{pmatrix} = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$$

be any 2×2 minor of B where $i \neq j$ for $i, j \geq 3$.

Our aim is to show that the only nonzero minors M_{ij} of B are those that have precisely one nonzero column whose entries are linearly independent. Suppose that B contains a minor M_{ij} with $a, d \neq 0$ and $c = 0$ or $c \notin \mathrm{span}(a, d)$. Define a linear function f on the vector space $\mathrm{span}(a, d, c)$ such that $f(a) = 1, f(d) \neq 0$ and $f(c) = 0$. f can be trivially extended to a linear function $f \in \mathfrak{g}^*$. Then $\{e_1, e_2, e_i\}$ are linearly independent over $\mathfrak{g}^f$, which implies that the index of $\mathfrak{g}$ is less than $n-2$ which contradicts our hypothesis. The independence of those three elements can be easily checked, since if $x = \alpha e_1 + \beta e_2 + \gamma e_i \in \mathfrak{g}^f$, then $0 = f([x, e_i]) = \alpha f(a) + \beta f(c) = \alpha$ implying $\alpha = 0$. Analogously $0 = f([x, e_j]) = \beta f(d)$ implies $\beta = 0$ and $0 = f([x, e_1]) = \gamma f(a)$ shows $\gamma = 0$. Thus B cannot contain a minor of the given form.

In particular if B contains any nonzero column whose entries are linearly dependent, say $[e_2, e_i] = \lambda[e_1, e_i]$ for some $i \geq 3$ and $\lambda \neq 0$, then after the base change replacing e_2 with $e_2' = e_2 - \lambda e_1$, we obtain $[e_2', e_i] = 0$ and $[e_1, e_i] \neq 0$. If there existed any other column j such that $[e_2', e_j]$ is nonzero, then we would have a minor M_{ij} of an impossible shape. Hence $[e_2', e_j] = 0$ for all $j \geq 3$. However this means that $\mathfrak{a} \oplus \mathbb{C}e_2'$ is an abelian ideal of codimension 1 which contradicts our hypothesis. Thus we showed that the entries of any nonzero column of B are linearly independent. Moreover if two such nonzero columns existed, say at position i and j, then $c \in \operatorname{span}(a, d)$, for $a = [e_1, e_i], c = [e_2, e_i]$ and $d = [e_2, e_j]$, otherwise M_{ij} had an impossible shape. Since a and c are linearly independent $d \in \operatorname{span}(a, c)$ and there exist $\alpha, \beta \in \mathbb{C}$ such that $d = \alpha a + \beta c$. After the base change replacing e_j with $e_j' = e_j - \beta e_i$, we obtain $[e_2, e_j'] = d - \beta c = \alpha a$. Thus the minor M_{ij} has an impossible form, since c and a are linearly independent. We conclude that B has precisely one nonzero column. Without loss of generality we may assume that $[e_1, e_3] \neq 0$ and that we rearrange the basis of $\mathfrak{a}$ such that $[e_1, e_3] = e_4$, $[e_2, e_3] = e_5$ and $[e_1, e_i] = 0$ and $[e_2, e_i] = 0$ for all $i \geq 4$.

Since $[e_1, e_2] \in \mathfrak{a}$, there exist $\alpha, \beta, \gamma \in \mathbb{C}$ such that $[e_1, e_2] = \alpha e_3 + \beta e_4 + \gamma e_5 + y \in \mathfrak{a}$ for $y \in \langle e_6, \ldots, e_n \rangle$. We now consider the following two cases:

Case 1. Suppose that $y \neq 0$. Then $\{e_3, e_4, e_5, [e_1, e_2]\}$ is a linearly independent subset of $\mathfrak{a}$ and we can complete it to a basis $\{e_3, e_4, e_5, e_6', \ldots, e_n'\}$ of $\mathfrak{a}$ where $e_6' = [e_1, e_2]$. The nonzero brackets of $\mathfrak{g}$ are $[e_1, e_3] = e_4, [e_2, e_3] = e_5, [e_1, e_2] = e_6'$. Hence $\mathfrak{g}$ is the direct product $\mathfrak{g} = \mathfrak{h}_6 \times \mathfrak{a}'$, where $\mathfrak{a}' = \langle e_7', \ldots, e_n' \rangle$ is the $(n-6)$-dimensional abelian Lie algebra.

Case 2. If $y = 0$, then first note that $\alpha \neq 0$ because if $[e_1, e_2] = \beta e_4 + \gamma e_5$, then the base change replacing e_1 with $e_1' = e_1 + \gamma e_3$ and e_2 with $e_2' = e_2 - \beta e_3$ yields

$$[e_1', e_2'] = [e_1, e_2] - \beta[e_1, e_3] + \gamma[e_3, e_2] = \beta e_4 + \gamma e_5 - \beta e_4 - \gamma e_5 = 0.$$

So $\mathfrak{g}$ has an abelian ideal of codimension one, which contradicts our hypothesis. So α must be nonzero and we carry out the following base change replacing e_3 with $e_3' = \alpha e_3 + \beta e_4 + \gamma e_5$ as well as replacing e_4 with $e_4' = \alpha e_4$ and e_5 with $e_5' = \alpha e_5$. Hence

$$[e_1, e_2] = e_3', \qquad [e_1, e_3'] = e_4', \qquad [e_2, e_3'] = e_5'.$$

Thus $\mathfrak{g}$ is the direct product $\mathfrak{g} = \mathfrak{h}_5 \times \mathfrak{a}'$, where $\mathfrak{a}' = \langle e_6, \dots, e_n \rangle$ is the $(n-5)$-dimensional abelian Lie algebra.

($\Leftarrow$) Now we prove the converse. First let $\mathfrak{g}$ be a finite dimensional Lie algebra with an abelian ideal $\mathfrak{a}$ of codimension one and write $\mathfrak{g} = ke \oplus \mathfrak{a}$. Take any $f \in \mathfrak{g}^*$ such that $\mathfrak{g}^f \neq \mathfrak{g}$. Then there exist $x, y \in \mathfrak{g}$ such that $f([x,y]) \neq 0$. If we write $x = \lambda e + a$ and $y = \mu e + b$ for some $a, b \in \mathfrak{a}$ and $\lambda, \mu \in k$, then we have $0 \neq f([x,y]) = f([e, \lambda b - \mu a])$. Hence $e \notin \mathfrak{g}^f$ and we might assume that $x = e$ and $y \in \mathfrak{a}$. For any other element $z \in \mathfrak{a} \backslash \mathfrak{g}^f$ we have $f([e,z]) \neq 0$ otherwise $z \in \mathfrak{g}^f$. Hence

$$f([e, f([e,y])z - f([e,z])y]) = f([e,y])f([e,z]) - f([e,z])f([e,y]) = 0$$

shows that $f([e,y])z - f([e,z])y \in \mathfrak{g}^f$, *i.e.* z and y are linearly dependent over $\mathfrak{g}^f$, hence $\dim(\mathfrak{g}/\mathfrak{g}^f) = 2$.

Now suppose that $\mathfrak{g}$ is isomorphic to $\mathfrak{h}_5$. Note that e_4 and e_5 are central and so they belong to $\mathfrak{h}_5^f$ for any f. Hence $\mathrm{ind}(\mathfrak{g})$ is at least 2. Since the number $\dim(\mathfrak{g}) - \mathrm{ind}(\mathfrak{g})$ is always even, we must have $\mathrm{ind}(\mathfrak{g}) = 3$.

Lastly we handle the case $\mathfrak{h}_6$ in a similar way. In this case the basis elements e_4, e_5, e_6 are central. Hence for any functional f these elements belong to the space $\mathfrak{h}_6^f$. Hence the index is at least 3. Again, since the space $\mathfrak{g}/\mathfrak{g}^f$ is even dimensional, it follows that the index is at least 4. □

Now we are set to prove the main theorem.

Proof:[Proof of the Main Theorem 3.1.1] $(a) \Leftrightarrow (c)$ and$(b) \Leftrightarrow (c)$ follow from Proposition 3.5.3. $(c) \Leftrightarrow (d)$ follows from Proposition 3.6.3. □

3.7 Examples

Finite dimensional nilpotent Lie algebras $\mathfrak{g}$ with an abelian ideal of codimension 1 are in bijection with finite dimensional vector spaces V and nilpotent endomorphisms $f : V \to V$. For such data one defines $\mathfrak{g} = \mathbb{C}e \oplus V$ and $[e,x] = f(x)$ for all $x \in V$. An example of this construction is given by the n-dimensional **standard filiform** Lie algebra, which is the Lie algebra on the vector space $\mathcal{L}_n = \mathrm{span}\{e_1,\dots,e_n\}$ such that the only nonzero brackets are given by $[e_1,e_i] = e_{i+1}$ for all $2 \le i < n$. Hence $\mathcal{L}_n$ provides an example of a non-abelian nilpotent Lie algebra $\mathfrak{g}$ such that $U(\mathfrak{g})$ has property $(\diamond)$. The 3-dimensional Heisenberg Lie algebra occurs as $\mathcal{L}_3$.

Given an even dimensional complex vector space $V = \mathbb{C}^{2n}$ and an anti-symmetric bilinear form $\omega : V \times V \to \mathbb{C}$, one defines the $2n+1$-dimensional **Heisenberg Lie algebra** associated to (V,ω) as $\mathcal{H}_{2n+1} = V \oplus \mathbb{C}h$ with h being central and $[x,y] = \omega(x,y)h$ for all $x,y \in V$. Note that $\mathrm{ind}\mathcal{H}_{2n+1} = 1$. Thus $U(\mathcal{H}_{2n+1})$ has property $(\diamond)$ if and only if $n = 1$, *i.e.* for $\mathcal{H}_3 = \mathcal{L}_3$.

In [56] a finite dimensional Lie superalgebra $\mathfrak{g}$ is called a **Heisenberg Lie superalgebra** if it has a 1-dimensional homogeneous center $\mathbb{C}h = Z(\mathfrak{g})$ such that $[\mathfrak{g},\mathfrak{g}] \subseteq Z(\mathfrak{g})$ and such that the associated homogeneous skew-supersymmetric bilinear form $\omega : \mathfrak{g} \times \mathfrak{g} \to \mathbb{C}$ given by $[x,y] = \omega(x,y)h$ for all $x,y \in \mathfrak{g}$ is non-degenerated when extended to $\mathfrak{g}/Z(\mathfrak{g})$. On the other hand one can construct a Heisenberg Lie superalgebra on any finite-dimensional supersymplectic vector superspace V with a homogeneous supersymplectic form ω.

By [56, page 73] if ω is even, *i.e.* $\omega(\mathfrak{g}_0,\mathfrak{g}_1) = 0$, then $\mathfrak{g}_0$ is a Heisenberg Lie algebra and if ω is odd, *i.e.* $\omega(\mathfrak{g}_i,\mathfrak{g}_i) = 0$ for $i \in \{0,1\}$, then $\mathfrak{g}_0$ is abelian. Hence $U(\mathfrak{g})$ has property $(\diamond)$ if and only if ω is odd or $\dim \mathfrak{g}_0 \le 3$.

Chapter 4

Differential operator rings

In this chapter we consider property $(\diamond)$ for differential operator rings. We give a complete answer to the question when $S = k[x][y; p\frac{\partial}{\partial x}]$ has property $(\diamond)$ for a field k of characteristic zero. We achieve this by showing that there exists a non-Artinian finitely generated essential extension of a simple S-module if and only if d is not locally nilpotent or equivalently if and only if S is not isomorphic to neither the polynomial ring $k[x, y]$ nor the first Weyl algebra $A_1(k)$. Combining this characterization with a result of Carvalho and Musson [9] and results of Alev and Dumas we are also able to characterize Ore extensions of $k[x]$ which have property $(\diamond)$.

Among examples of Noetherian rings which do not have property $(\diamond)$, we have presented Musson's example of an enveloping algebra of a finite dimensional Lie algebra in § 2.3.2. When $n = 1$ Musson's example becomes the differential operator ring $k[x][y; x\frac{\partial}{\partial x}]$. Recently, Musson extended this example and proved that the differential operator ring $R = k[x][y; x^r\frac{\partial}{\partial x}]$, $r \in \mathbb{N}$, does not have property $(\diamond)$ [50]. He also presented sufficient conditions for Noetherian algebras over a field for the existence of non-Artinian essential extensions of simple modules over such algebras. We follow a different method and show that for any nonconstant polynomial p, $k[x][y; p\frac{\partial}{\partial x}]$ does not have property $(\diamond)$.

We start in the first section with some general results on differential operator rings over commutative domains and we give a method to construct non-Artinian essential extensions of some simple modules under certain assumptions. Next we consider Ore extensions of the polynomial ring in one variable and obtain a characterization of such extensions which have property ($\diamond$). In the last part we consider differential operator rings over commutative Noetherian domains and show that a differential operator ring with a locally nilpotent derivation d over a commutative finitely generated algebra R has property ($\diamond$).

4.1 General results on differential operator rings over commutative domains

In this section we assume that k is a field, R is a k-algebra, d is a k-linear derivation of R and $S = R[y;d]$ is the differential operator ring defined by d. That is, S is an overring of R which is also free as a left R-module with basis $\{y^n \mid n \geq 0\}$, and whose multiplication is subject to the relation $ya = ay + d(a)$ for all $a \in R$. Moreover, S is also free as a right R-module with the same basis and the following relations hold

$$y^n a = \sum_{i=0}^{n} \binom{n}{i} d^{n-i}(a) y^i \quad \text{and} \quad a y^n = \sum_{i=0}^{n} \binom{n}{i} (-1)^i y^{n-i} d^i(a) \ \forall a \in R,\ n \geq 0.$$

A subset I of R is called **d-stable** if $d(I) \subseteq I$. An ideal of R which is d-stable is called a **d-ideal**. In order to show that S does not have property ($\diamond$) we will construct a simple left S-module E and a cyclic essential extension M of E with M/E being non-Artinian. A suitable construction of a simple left R-module is given by the following proposition which also follows from a result by Goodearl and Warfield (see [22, Proposition 3.1]). However for the sake of completeness, we will include a proof of this fact here.

Proposition 4.1.1 *Let R be a commutative k-algebra with $\mathrm{char}(k) = 0$ and let $S = R[y;d]$ for some derivation d of R. If $\mathfrak{m}$ is a maximal ideal of R that is not d-stable, then $S\mathfrak{m}$ is a maximal left ideal of S.*

Proof: Since $\mathfrak{m}$ is not d-stable, there exists a nonzero element $a \in \mathfrak{m}$ such that $d(a) \notin \mathfrak{m}$. Write $v = 1 + S\mathfrak{m}$ for the canonical generator of $E = S/S\mathfrak{m}$. We first prove by induction on n that the following statement holds for any $n > 0$

$$(a^n y^n)v = c_n v \qquad \text{where} \qquad c_n = (n!)(-d(a))^n \in R \setminus \mathfrak{m}.$$

For $n = 1$ we have

$$ayv = yav - d(a)v = c_1 v$$

since $ya \in S\mathfrak{m}$. Suppose that $n > 0$ and $(a^n y^n)v = c_n v$ holds with $c_n \in R \setminus \mathfrak{m}$ as above, then

$$\begin{aligned} a^{n+1}y^{n+1}v &= (ya^{n+1} - d(a^{n+1}))y^n v \\ &= (ya - (n+1)d(a))a^n y^n v \\ &= (ya - (n+1)d(a))c_n v \\ &= -(n+1)d(a)c_n v = c_{n+1}v. \end{aligned}$$

as $yac_n \in S\mathfrak{m}$. Moreover $c_{n+1} = -(n+1)d(a)c_n \notin \mathfrak{m}$ as $d(a),\ c_n \notin \mathfrak{m}$ and char(k) is zero. Hence we proved our claim.

Let $f \in Sv$ be a nonzero element and suppose that f is represented as $f = \sum_{i=0}^{n} b_i y^i v$, with n being minimal. Then $b_n \in R \setminus \mathfrak{m}$. Since $R/\mathfrak{m}$ is a field and $b_n, c_n \notin \mathfrak{m}$, there exists $u \in R$ such that $ub_nc_n - 1 \in S\mathfrak{m}$. Hence $ua^n f = ub_nc_nv = v$ shows that E is a simple left S-module. □

Let $\mathfrak{m}$ be a maximal ideal of R which is not d-stable. We would like to build an essential extension of the simple left S-module $S/S\mathfrak{m}$ which is not Artinian. We need the following lemma in order to do this.

Lemma 4.1.2 *Let R and S be as in Proposition 4.1.1. Then $S = R \oplus S(y-1)$ as k-vector spaces.*

Proof: Since $\left(\sum_{i=0}^{n-1} y^i\right)(y-1) = y^n - 1$ holds for all $n > 0$, we have for all $a \in R$:

$$ay^n = a + \left(\sum_{i=0}^{n-1} ay^i\right)(y-1) \in R + S(y-1).$$

Moreover if $a = \sum_{i=0}^{n} b_i y^i (y-1) \in R \cap S(y-1)$ with $b_i \in R$, then bearing in mind that the powers of y form a basis of S as left R-module and comparing coefficients, we see that $b_i = 0$ for all i, *i.e.* $a = 0$. □

We call the ring R left **d-simple** if its only d-ideals are 0 and R. In the following result we use a lattice isomorphism between the d-ideals of R and the lattice of S-submodules of $S/S(y-1)$, and therefore obtain a condition to decide when the S-module $S/S(y-1)$ is Artinian.

Proposition 4.1.3 *Let R be a commutative Noetherian domain and $S = R[y; d]$ for some derivation d of R. Then $S/S(y-1)$ is Artinian as left S-module if and only if R is d-simple.*

Proof: Let $\pi : S \to S/S(y-1)$ be the canonical projection. There exists a lattice isomorphism between the d-ideals I of R and the left S-submodules of $S/S(y-1)$ given as follows: for any d-ideal I of R, $\pi(I)$ is a left S-submodule, because for any $a \in I$:

$$y\pi(a) = \pi(ay + d(a)) = \pi(a + d(a)) \in \pi(I).$$

Moreover if U is a left S-submodule of $S/S(y-1)$, then $I = \pi^{-1}(U) \cap R$ is a d-ideal of R since for any $a \in I$ we have

$$\pi(d(a)) = \pi(ya - ay) = \pi((y-1)a) \in U,$$

i.e. $d(a) \in I$. Because of Lemma 4.1.2 it is clear that $U = \pi(I)$.

Suppose that $S/S(y-1)$ is Artinian as left S-module. By the lattice theoretical correspondence R satisfies the descending chain condition for d-ideals. Hence given a proper d-ideal I of R, the chain $I \supseteq I^2 \supseteq \cdots$ must stop and there exists $k \geq 1$ such that $I^k = I^{k+1}$. Therefore $I^k = (I^k)^2$ and I^k is an idempotent ideal. Since any idempotent ideal of a commutative Noetherian domain is trivial, $I^k = 0$ or $I^k = R$. As R is a domain, $I = 0$ or $I = R$. The converse is clear, since by the lattice theoretical isomorphism $S/S(y-1)$ is a simple left S-module if R is d-simple. □

Proposition 4.1.1 and Proposition 4.1.3 show that if R is a commutative Noetherian domain which is not d-simple and has a maximal ideal $\mathfrak{m}$ which is not d-stable, then

$S/S\mathfrak{m}(y-1)$ is a cyclic left S-module which is an extension of the simple left S-module $S/S\mathfrak{m} \simeq S(y-1)/S\mathfrak{m}(y-1)$. The last proposition of this section gives a sufficient condition to assure the essentiality of this extension. An element $a \in R$ is called a **Darboux element** with respect to d if $d(a) = ba$ for some $b \in R$. If the context is clear we simply refer to a as a Darboux element without mentioning the derivation d. In other words a is a Darboux element if and only if Ra is a d-ideal of R. If R is commutative, then $a \in R$ is a Darboux element if and only if a is a normal element in S. In fact if a is a Darboux element, then $ya = ay + d(a) = a(y+b)$. Hence $y^n a = a(y+b)^n$ and also $ay^n = (y-b)^n a$, showing $Sa = aS$. On the other hand if a is normal in S, then $ay \in Sa$. Thus there exists $g \in S$ such that $ga = ay = ya - d(a)$. Looking at the zero component of both sides and taking into account that S is free as right R-module, there exists $b \in R$ such that $d(a) = ba$. Recall that a normal element a in a domain S induces an automorphism σ of S defined by $ra = a\sigma(r)$, for all $r \in S$.

Proposition 4.1.4 *Let R be a commutative Noetherian domain which is also an algebra over a field k of characteristic zero. Let d be a derivation of R and set $S = R[y; d]$. Suppose that the following conditions hold:*

(1) R is not d-simple;

(2) there exists a maximal ideal $\mathfrak{m}$ of R that does not contain any nonzero d-ideal;

(3) every nonzero d-ideal contains a nonzero Darboux element.

Then $S/S\mathfrak{m}(y-1)$ is a non-Artinian essential extension of the simple left S-module $S/S\mathfrak{m}$, i.e. S does not have property $(\diamond)$.

Proof: For simplicity, set $L = S\mathfrak{m}(y-1)$ and $M = S/L$. By Proposition 4.1.1

$$E := S(y-1)/L \simeq S/S\mathfrak{m} \tag{4.1}$$

is a simple left S-module since $\mathfrak{m}$ is not d-stable. By Proposition 4.1.3 the left S-module $M/E = S/S(y-1)$ is not Artinian, since S is not d-simple. We need to show that M is an essential extension of E. Write $\pi : S \to S/L$ for the canonical projection. Let U be a nonzero left S-submodule of M and suppose that $U \neq E$. Recall that $S = R \oplus S(y-1)$ by Lemma 4.1.2 and set

$$I = \{a \in R \mid \exists g \in S : \pi(a + g(y-1)) \in U\}. \tag{4.2}$$

Since $(y-1)a = a(y-1) - d(a)$ for any $a \in R$, we see that I is a d-ideal of R. I is nonzero as $U \neq E$. By hypothesis I contains a nonzero Darboux element $f \in I$. Let $q \in R$ such that $d(f) = qf$ and let $g \in S$ be such that $\pi(f + g(y-1)) \in U$. In particular $(y-q)f = fy + d(f) - qf = fy$. Then

$$(y-1-q)\pi(\,f + g(y-1)\,) = \pi(\,(f + (y-1-q)g)(y-1)\,) \in E \cap U \tag{4.3}$$

We will show that $\pi(\,(f + (y-1-q)g)(y-1)\,) \neq 0$, in order to conclude that E is essential in M. Thus suppose

$$(f + (y-1-q)g)(y-1) \in L = S\mathfrak{m}(y-1). \tag{4.4}$$

As S is a domain, this is equivalent to

$$f + (y-1-q)g \in S\mathfrak{m}. \tag{4.5}$$

Note that f is a normal element in S and as S is a domain, there exists an automorphism σ of S with $fr = \sigma(r)f$ for all $r \in R$. Since $fy = (y-q)f$ we have $\sigma(y) = y - q$.

Let $R = \mathfrak{m} \oplus V$ be a vector space decomposition of R with $1 \in V$ and denote by a_V the V-component of an element $a \in R$. Since S is a free right R-module with basis $\{y^i \mid i \geq 0\}$, we have the vector space decomposition $S = S\mathfrak{m} \oplus \bigoplus_{i=0}^{\infty} y^i V$. Hence there exist $b \in S\mathfrak{m}$ and $v_0, \ldots, v_m \in V$ such that $\sigma^{-1}(g) = b + \sum_{i=0}^{m} y^i v_i$. Since $\sigma(a) = a$ for any $a \in R$ and taking into account that $\sigma(b) \in \sigma(S\mathfrak{m}) = S\mathfrak{m}$, we have

$$f_V + \sigma(y-1)g - \sigma((y-1)b) = \sigma(f_V + (y-1)(\sigma^{-1}(g) - b)) = \sigma\left(f_V + (y-1)\sum_{i=0}^{m} y^i v_i\right) \tag{4.6}$$

Hence the left hand side belongs to $S\mathfrak{m} = \sigma(S\mathfrak{m})$ while the right hand side belongs to $\sigma\left(\bigoplus_{i=0}^{\infty} y^i V\right)$. As

$$\sigma(S\mathfrak{m}) \cap \sigma\left(\bigoplus_{i=0}^{\infty} y^i V\right) = \sigma\left(S\mathfrak{m} \cap \bigoplus_{i=0}^{\infty} y^i V\right) = 0 \tag{4.7}$$

and as σ is an automorphism we have

$$f_V + (y-1)\sum_{i=0}^{m} y^i v_i = y^{m+1} v_m + \sum_{i=1}^{m} y^i (v_{i-1} - v_i) + f_V - v_0 = 0 \tag{4.8}$$

Hence $v_i = 0$ for all i as well as $f_V = 0$, which implies that $f \in \mathfrak{m}$, which induces a non-trivial d-ideal in $\mathfrak{m}$, contradicting the hypothesis. □

Following Goodearl and Warfield [22], we call the rings R which satisfy condition (2) of Proposition 4.1.4 **d-primitive**.

4.2 Ore extensions of $K[x]$

In this section we apply the general results of the previous section to $R = k[x]$. First we remark a few facts on derivations of polynomial rings. Let d be a derivation of R and let $d(x) = p \in R$. By definition, $d(a) = 0$ for every $a \in k$. Moreover, it is not difficult to see that

$$d(x^i) = ix^{i-1}d(x) = ix^{i-1}p$$

for every $i \geq 1$. For an arbitrary element $\sum_{i=0}^{n} \lambda_i x^i$ of R we have

$$d\left(\sum_{i=0}^{n} \lambda_i x^i\right) = \sum_{i=0}^{n} (d(\lambda_i)x^i + \lambda_i d(x^i)) = \sum_{i=0}^{n} \lambda_i i x^{i-1} p = p\frac{\partial}{\partial x}\left(\sum_{i=0}^{n} \lambda_i x^i\right).$$

So that $d = p\frac{\partial}{\partial x}$. Conversely, for any polynomial $p \in R$, $d = p\frac{\partial}{\partial x}$ defines a derivation of R. Hence any derivation d of R is completely determined by $d(x) = p \in R$.

Corollary 4.2.1 *Let* $\mathrm{char}(k) = 0$, $p \in k[x]$, $d = p\frac{\partial}{\partial x}$ *and* $S = k[x][y; d]$. *The following statements are equivalent:*

(a) S *has property* $(\diamond)$.

(b) p *is a constant polynomial.*

(c) $S \simeq k[x, y]$ *or* $S \simeq A_1(k)$.

(d) S *is commutative or has Krull dimension* 1.

Proof: Let $R = k[x]$. First note that any nonzero d-ideal I of R contains a Darboux element, because as R is a principal ideal domain, any of the generators of I is a nonzero Darboux element. Let $\alpha \in k$ be any element that is not a root of p (which exists as k is infinite). Then $\mathfrak{m} = \langle x - \alpha \rangle$ is a maximal ideal which does not contain a nonzero Darboux element, because if such an element belonged to $\mathfrak{m}$, say $f \in \mathfrak{m}$, then there would exist $h \in R$ and $n > 0$ such that $f = h(x-\alpha)^n$ and $(x-\alpha) \nmid h$. Suppose that $d(f) = gf$ for some $g \in R$, then

$$gh(x-\alpha)^n = gf = d(f) = d(h)(x-\alpha)^n + nhp(x-\alpha)^{n-1} = (d(h)(x-\alpha) - nhp)(x-\alpha)^{n-1}$$

which would imply $(gh - d(h))(x-\alpha) = -nhp$, *i.e.* $(x-\alpha) \mid h$ a contradiction.

$(a) \Rightarrow (b)$ If p is not constant, then $I = Rp$ is a nonzero d-ideal, *i.e.* R is not d-simple. By Proposition 4.1.4 S does not have property $(\diamond)$.

$(b) \Rightarrow (c)$ If p is constant, then $S \simeq k[x,y]$ if $p = 0$ and $S \simeq A_1(K)$ if $p \neq 0$. It is clear that $(c) \Rightarrow (d)$.

Finally, $(d) \Rightarrow (a)$ follows because commutative Noetherian domains and semiprime Noetherian rings of Krull dimension one have property $(\diamond)$. □

A result of Awami, Van den Bergh, Van Oystaeyen and of Alev and Dumas states that given an automorphism σ of $K[x]$ and a σ-derivation δ, the Ore extension $S = K[x][y;\sigma,\delta]$ falls, upto isomorphism, into four classes of rings, as we record in the following:

Proposition 4.2.2 *[2, (2.1)][1, (3.2)] Let k be a field, and $R = k[x]$ be the polynomial algebra over k in the commuting variable x. If σ is an automorphism of R and if δ is a σ-derivation of R, then the resulting Ore extension $S = k[x][y;\sigma,\delta]$ is isomorphic to one of the following algebras:*

(a) $S = k[x,y]$ *is commutative.*

(b) $S = k_q[x,y]$ *is the quantum plane for some* $q \in k$.

(c) $S = A_1^q(k)$ *is the quantum Weyl algebra for some* $q \in k$.

(d) $S = k[x][y;\delta]$ *is a differential operator ring.*

Since any automorphism of $k[x]$ is such that $\sigma(x) = qx + b$ for some $q, b \in k$, following the proof of [2] and [1], we have the following possibilities:

Case 1. If $q \neq 1$, then $S \simeq k[x'][y;\sigma',\delta]$ where $x' = x + b(q-1)^{-1}$ and $\sigma'(x') = qx'$. Now if $p \in k[x]$ and $r \in k$ are such that $\delta(x') = p(x')(1-q)x' + r$ then

$$(y + p(x'))x' = qx'(y + p(x')) + r.$$

If $r = 0$, it is easy to see that $S \simeq k_q[x', y'] = k\langle x', y' \mid y'x' = qx'y' \rangle$ for a suitable change of variables. If $r \neq 0$, taking $y'' = r^{-1}(y+p)$, $S \simeq A_1^q(k) = k\langle x', y'' \mid y''x' = qx'y'' + 1 \rangle$.

Case 2. If $q = 1$ and $b = 0$, S is either $k[x,y]$ or a differential operator ring, $S = k[x][y;\delta]$.

Case 3. If $q = 1$ and $b \neq 0$, $S \simeq R[x'][y;\sigma',\delta]$ by making $x' = b^{-1}x$, $\sigma'(x') = x' + 1$ and $\delta'(x') = b^{-1}\delta(x)$. Since

$$(y + \delta(x))x' = (x' + 1)(y + \delta(x))$$

it follows that $S \simeq k[y'][x'; -y'\frac{\partial}{\partial y'}]$.

Using this characterisation we can determine precisely when $k[x][y;\sigma,d]$ has property $(\diamond)$:

Theorem 4.2.3 *Let k be a field of characteristic zero and let σ be an automorphism of $k[x]$ and d be a σ-derivation of $k[x]$. Let $q, b \in k$ such that $\sigma(x) = qx + b$. The following statements are equivalent:*

(a) $S = k[x][y;\sigma,d]$ *has property* $(\diamond)$*.*

(b) $S \simeq k_q[x,y]$ *or* $S \simeq A_1^q(K)$ *for q a root of unity (including $q = 1$).*

(c) $q \neq 1$ *is a root of unity or* $q = 1$ *and $d(x)$ is a constant polynomial.*

(d) $\sigma \neq id$ *has finite order or* $\sigma = id$ *and d is locally nilpotent.*

Proof: By Corollary 4.2.1 any algebra of the form $k[x][y;d]$ having property $(\diamond)$ has to be isomorphic to the first Weyl algebra or to the polynomial ring, *i.e.* to $A_1^q(k)$ or to $k_q[x,y]$ for $q = 1$. If q is not a root of unity, then [9, Theorem 3.1] respectively [9, Theorem 4.2] shows that $k_q[x,y]$ respectively $A_1^q(k)$ does not have property $(\diamond)$. On the other hand if $q \neq 1$ is a root of unity, then $k_q[x,y]$ and $A_1^q(k)$ are PI-algebras and hence in particular FBN which have property $(\diamond)$ by Jategaonkar's result (see [33, Corollary 3.6]). The case $q = 1$ is obtained from the fact that the first Weyl algebra is a prime Noetherian algebra of Krull dimension 1. Together with the characterisation above, this shows $(a) \Leftrightarrow (b) \Leftrightarrow (c)$.

$(c) \Leftrightarrow (d)$ Note that for all $n > 1$ we have $\sigma^n(x) = q^n x + \frac{q^n-1}{q-1} b$ if $q \neq 1$. Suppose $\sigma \neq id$, then σ has order n if and only if q is an nth root of unity. □

4.3 Differential operator rings over commutative Noetherian domains

Let k be an algebraically closed field of characteristic zero, let $R = k[x_1,\ldots,x_n]$ be the polynomial ring in n variables over k and let $d = p_1\frac{\partial}{\partial x_1} + \ldots + p_n\frac{\partial}{\partial x_n}$ be a nonzero derivation of R for some polynomials $p_1,\ldots,p_n \in R$. Set $S = R[y;d]$. Following Moulin-Ollagnier and Nowicki (see [54]) we call d **irreducible** if $I = \langle p_1,\ldots,p_n\rangle = R$. Suppose that d is not irreducible, then there exists a point $\alpha = (\alpha_1,\ldots,\alpha_n) \in k^n$ that does not belong to the algebraic variety $\mathcal{V}(I)$ defined by I. Thus we have that the ideal $\mathfrak{m} = \langle x_1-\alpha_1,\ldots,x_n-\alpha_n\rangle$ of R generated by $x_1-\alpha_1,\ldots,x_n-\alpha_n$ is a maximal ideal of R that is not d-stable, because otherwise $d(x_i-\alpha_i) = p_i \in \mathfrak{m}$ for all i, which implies $I \subseteq \mathfrak{m}$ and hence $\alpha \in \mathcal{V}(\mathfrak{m}) \subseteq \mathcal{V}(I)$ contradicting the choice of α. Thus $S/S\mathfrak{m}$ is a simple left S-module by Proposition 4.1.1 and $S/S(y-1)$ is not Artinian by Proposition 4.1.3. If $\mathfrak{m}$ does not contain any nonzero d-ideal and every nonzero d-ideal of R contains nonzero Darboux elements, then we can conclude by Proposition 4.1.4 that $S = R[y;d]$ does not have property $(\diamond)$. However neither do we know of good criteria to secure the existence of Darboux elements in d-ideals of $R = k[x_1,\ldots,x_n]$ nor whether $\mathfrak{m}$ could contain nonzero Darboux elements.

A stronger assumption than assuming the existence of Darboux elements in nonzero

d-ideals, is to assume that any nonzero d-ideal I of R intersects non-trivially the subring of constants $R^d = \{a \in R \mid d(a) = 0\}$. This is the case for a locally nilpotent derivation d. We saw in Theorem 4.2.3 that an Ore extension of $k[x]$ by some automorphism σ and σ-derivation d has property $(\diamond)$ if and only if $\sigma \neq id$ has finite order or $\sigma = id$ and d is locally nilpotent. If R is a Noetherian commutative domain, d is a σ-derivation and σ is of finite order, then $R[y;\sigma,d]$ is a PI-algebra by [42, Theorem 4], hence has property $(\diamond)$.

We conclude this chapter with a general statement showing that for the case $\sigma = id$ a differential operator ring with a locally nilpotent derivation d over a commutative finitely generated algebra R also has property $(\diamond)$, using our result on the enveloping algebra of a finite dimensional nilpotent Lie algebra from the previous chapter.

Proposition 4.3.1 *Let k be an algebraically closed field of characteristic zero and let R be a commutative finitely generated k-algebra with locally nilpotent derivation d. Then all injective hulls of simple $R[y;d]$-modules are locally Artinian. Moreover, if R is a domain which is not a field, then either $R[y;d]$ is a simple ring or any maximal ideal of R intersects R^d non-trivially.*

Proof: Let $x_1,\dots,x_n$ be the algebra generators of R and consider the set

$$V = \left\{d^i(x_j) \mid 1 \leq j \leq n, i \geq 0\right\}.$$

Since d is locally nilpotent, V is a finite set containing all generators $x_1,\dots,x_n$. Let $\mathfrak{h} = \mathrm{span}(V)$ be the (finite dimensional) subspace of R generated by V. Consider $\mathfrak{g} = \mathfrak{h} \oplus ky$, which is a subspace of $S = R[y;d]$. Since $[d^i(x_j), y] = d^{i+1}(x_j) \in \mathfrak{h}$, the space $\mathfrak{g}$ is closed under the commutator bracket $[,]$ in S and hence is a Lie subalgebra of $(S,[,])$. Since d is locally nilpotent, $\mathfrak{g}$ is a (finite dimensional) nilpotent Lie algebra over k with the Abelian ideal $\mathfrak{h}$ of codimension 1. By Theorem 3.1.1 $U(\mathfrak{g})$ has property $(\diamond)$. The Lie algebra inclusion $\mathfrak{g} \to R[y;d]$ induces an algebra map $U(\mathfrak{g}) \to R[y;d]$ which is surjective, since $\mathfrak{g}$ contains y and all algebra generators of R. Thus $R[y;d]$ also has property $(\diamond)$.

Suppose R is a Noetherian domain which is not a field. If R is d-simple, then R^d is a field. If $\dim_{R^d}(R)$ were finite dimensional, then R would be an Artinian domain and hence a field. Hence R has infinite dimension over R^d. By [22, Theorem 2.3] $R[y;d]$

is simple. Thus if $R[y;d]$ is not simple, then R cannot be d-simple. Since d is locally nilpotent, it follows that every nonzero d-ideal of R intersects R^d non-trivially. Hence if there existed a maximal ideal $\mathfrak{m}$ of R that intersected R^d trivially, then $\mathfrak{m}$ could not contain a nonzero d-ideal. Thus by Proposition 4.1.4 $R[y;d]$ would not have property $(\diamond)$ which is a contradiction to what we just proved. Hence any maximal ideal of R must intersect R^d non-trivially. $\square$

There are finitely generated noncommutative Noetherian domains that have property $(\diamond)$ but for which Proposition 4.3.1 fails. As an example take $R = A_1(\mathbb{C})[x]$, which has property $(\diamond)$, because any maximal ideal $\mathfrak{m}$ of the centre of R is of the form $\mathfrak{m} = \langle x - \lambda \rangle$, with $\lambda \in \mathbb{C}$; the quotient ring $R/\mathfrak{m} \simeq A_1(\mathbb{C})$ does have property $(\diamond)$ and thus by [10, Proposition 1.6], R has property $(\diamond)$. On the other hand $S = R[y; \frac{\partial}{\partial x}] \simeq A_2(\mathbb{C})$ does not have property $(\diamond)$ by Stafford's result in [63], although $\frac{\partial}{\partial x}$ is a locally nilpotent derivation of R.

Chapter 5

Stable torsion theories

5.1 Generalities on torsion theories

Let R be an arbitrary associative ring with unity and let R-mod denote the category of left R-modules. First defined by Dickson in [15], a **torsion theory** τ on R-mod is a pair (T_τ, F_τ) of classes of left R-modules, satisfying the following properties:

(i) $T_\tau \cap F_\tau = 0$,

(ii) T_τ is closed under homomorphic images,

(iii) F_τ is closed under submodules,

(iv) For each M in R-mod, there exist $F \in F_\tau$ and $T \in T_\tau$ such that $M/T \cong F$.

T_τ is called the class of τ-**torsion** modules and F_τ is called the class of τ-**torsionfree** modules. Our main references for torsion theories are [20] and [64].

Let $\mathcal{A}$ and $\mathcal{B}$ be nonempty classes of left R-modules. If $\mathcal{A} = \{M \mid \mathrm{Hom}_R(M,N) = 0 \text{ for all } N \in \mathcal{B}\}$ then $\mathcal{A}$ is said to be the **left orthogonal complement** of $\mathcal{B}$. Similarly, if $\mathcal{B} = \{N \mid \mathrm{Hom}_R(M,N) = 0 \text{ for all } M \in \mathcal{A}\}$ then $\mathcal{B}$ is said to be the **right orthogonal complement** of $\mathcal{A}$. We say that the pair $(\mathcal{A},\mathcal{B})$ is a **complementary pair** whenever $\mathcal{A}$ is the left orthogonal complement of $\mathcal{B}$ and $\mathcal{B}$ is the right orthogonal complement of $\mathcal{A}$. In particular, if τ is a torsion theory then (T_τ, F_τ) is a complementary pair and every such pair defines a torsion theory.

An immediate consequence of the definition is that the class of torsion modules for a torsion theory τ is closed under extension. For, if

$$0 \to A \to M \to B \to 0$$

is a short exact sequence of left R-modules with A, B are τ-torsion, then for any τ-torsionfree left R-module F, the corresponding short exact sequence

$$0 \to 0 = \mathrm{Hom}(B, F) \to \mathrm{Hom}(M, F) \to \mathrm{Hom}(A, F) = 0$$

implies that M is also τ-torsion. Similarly, the class of τ-torsionfree modules is closed under extensions too. Also, while it is not required in the definition of a torsion theory, we will be working with torsion theories such that the class of torsion modules is closed under submodules. Such torsion theories are called **hereditary**. For the rest of this chapter a torsion theory shall always mean a hereditary torsion theory.

Let $\mathcal{C}$ be a class of left R-modules. If $\mathcal{F}$ is a right orthogonal complement of $\mathcal{C}$ and $\mathcal{T}$ is a left orthogonal complement of $\mathcal{F}$, then the pair $(\mathcal{T}, \mathcal{F})$ is a torsion theory in R-mod, called the **torsion theory generated by** $\mathcal{C}$. If $\mathcal{T}$ is the left orthogonal complement of $\mathcal{C}$ and $\mathcal{F}$ is the right orthogonal complement of $\mathcal{T}$, then the pair $(\mathcal{T}, \mathcal{F})$ is a torsion theory, called the **torsion theory cogenerated by** $\mathcal{C}$.

5.1.1 The Goldie torsion theory

Let M be a left R-module. An element $m \in M$ is called a **singular element** of M if $\mathrm{Ann}_R(m)$ is an essential left ideal of R. The set $\mathrm{Sing}(M)$ of all singular elements of M is a submodule of M called the **singular submodule** of M. A module M is called **singular** if $\mathrm{Sing}(M) = M$ and it is called **nonsingular** if $\mathrm{Sing}(M) = 0$. The class $F_{\mathcal{G}}$ of all nonsingular left R-modules forms a torsionfree class for a hereditary torsion theory on mod-R. We call this the **Goldie torsion theory** and denote it by $\tau_{\mathcal{G}}$.

For any right R-module M, its Goldie torsion submodule is $t_{\mathcal{G}}(M) = \{m \in M \mid m + \mathrm{Sing}(M) \in \mathrm{Sing}(M/\mathrm{Sing}(M))\}$. Hence, Goldie's torsion class $T_{\mathcal{G}}$ is precisely the class of modules with essential singular submodule, and corresponding torsionfree class $F_{\mathcal{G}}$ is the class of nonsingular modules.

5.1.2 The Dickson torsion theory

Let $\mathcal{S}$ be a representative class of nonisomorphic simple left R-modules. Then the torsion theory $\mathfrak{D}$ generated by $\mathcal{S}$ is called the **Dickson torsion theory**. The class of $\mathfrak{D}$-torsionfree left R-modules is the right orthogonal complement of $\mathcal{S}$ while the class of $\mathfrak{D}$-torsion left R-modules are the left orthogonal complements of the class of $\mathfrak{D}$-torsionfree modules. In particular, every simple left R-module is $\mathfrak{D}$-torsion. Hence the class of all $\mathfrak{D}$-torsionfree modules consists of all soclefree left R-modules. Moreover, if $M \in T_{\mathfrak{D}}$ then M is an essential extension of its socle.

5.2 Stable torsion theories

A left R-module M is called **semi-Artinian** if for every submodule $N \neq M$, M/N has nonzero socle. We first show that the class of $\mathfrak{D}$-torsion left R-modules is exactly the class of semi-Artinian left R-modules.

Lemma 5.2.1 *A left R-module M is $\mathfrak{D}$-torsion if and only if it is semi-Artinian.*

Proof: ($\Leftarrow$) Let M be a semi-Artinian left R-module and F be a $\mathfrak{D}$-torsionfree left R-module. Then $\mathrm{soc}(F) = 0$ by definition. We show that $\mathrm{Hom}(M,F) = 0$. Every nonzero R-homomorphism $f \in \mathrm{Hom}(M,F)$ gives rise to an injective R-homomorphism $f' : M/\ker f \to F$. Let S be the socle of $M/\ker f$. Then $f'(S) \subseteq \mathrm{soc}(F) = 0$, hence $S = 0$. But this implies that $M = \ker f$. Hence $f = 0$.

($\Rightarrow$) Suppose that M is a $\mathfrak{D}$-torsion left R-module. Since the class of torsion modules is closed under factor modules, any nonzero factor module of M is also $\mathfrak{D}$-torsion and cannot be $\mathfrak{D}$-torsionfree. Hence M/N has nonzero socle for every submodule N of M and so M is semi-Artinian. □

Recall that a module has finite length if and only if it is both Noetherian and Artinian. In fact, we can still have finite length if the module is Noetherian and semi-Artinian. The following result is known in the literature, but we include a proof for completeness:

Lemma 5.2.2 *A left R-module M has finite length if and only if it is Noetherian and semi-Artinian.*

Proof: We only need to show the sufficiency. Suppose that M is a left R-module which is left Noetherian and semi-Artinian. Since it is semi-Artinian, it has nonzero socle, and so there exists a minimal submodule say S_1 of M. Similarly, the factor M/S_1 has a simple submodule S_2/S_1. This way we obtain an ascending chain of submodules of M

$$0 \leq S_1 \leq S_2 \leq \ldots \leq S_i \leq \ldots$$

such that S_i/S_{i-1} is simple. Since M is Noetherian, this chain stops after finitely many steps at M and so it is a composition series for M. Thus M has finite length. □

We remark that a locally Artinian module M is semi-Artinian. To see this, let $N \subseteq M$ be a submodule of a locally Artinian module M and let $m \in M \backslash N$. Then Rm is Artinian and $Rm/(Rm \cap N) \simeq (Rm+N)/N \subseteq M/N$ has a nonzero socle. Hence M is semi-Artinian.

A torsion theory τ on R-mod is called **stable** if its torsion class is closed under injective hulls. One of the equivalent conditions for $\mathcal{D}$ to be stable is that modules with essential socle are $\mathcal{D}$-torsion [15, 4.13]. Using this characterization, for a left Noetherian ring we obtain a connection between the stability of the Dickson torsion theory and property ($\diamond$) in the following result:

Proposition 5.2.3 *Let R be a left Noetherian ring. Injective hulls of simple left R-modules are locally Artinian if and only if the Dickson torsion theory is stable, if and only if the class of semi-Artinian left R-modules is closed under injective hulls.*

Proof: First we show that property ($\diamond$) implies the stability of $\mathcal{D}$. Let M be a $\mathcal{D}$-torsion left R-module. Then M has an essential socle and so that its injective hull $E(M)$ is a direct sum of injective hulls of simple left R-modules. Then $E(M)$ is locally Artinian by assumption. Since locally Artinian modules are semi-Artinian, it follows that $E(M)$ is $\mathcal{D}$-torsion and $\mathcal{D}$ is stable.

Conversely, let S be a simple left R-module and $E(S)$ be its injective hull. Let $0 \neq F \leq E(S)$ be a finitely generated submodule of $E(S)$. By assumption, $E(S)$ and hence F is $\mathcal{D}$-torsion and therefore semi-Artinian. Since F is Noetherian, by Lemma 5.2.2 F has finite length. Thus $E(S)$ is locally Artinian.

The last equivalence follows from Lemma 5.2.1. □

Hence the stability of the Dickson torsion theory is a necessary and sufficient condition for property $(\diamond)$. In [15, 4.13] Dickson characterizes those rings for which the Dickson torsion theory in the category R-Mod is stable. Indeed he considers the stability of the torsion theory generated by simple objects in any abelian category with injective envelopes. Translating his results to the language of our work, we list his characterization as follows:

Proposition 5.2.4 *The following are equivalent for a ring R.*

(i) The Dickson torsion theory is stable.

(ii) Any $\mathcal{D}$-torsion R-module can be embedded in a $\mathcal{D}$-torsion injective R-module.

(iii) Any injective R-module M decomposes as $M = M_t \oplus F$, where M_t is the torsion part of M and F is unique up to isomorphism and has zero socle.

(iv) If M is an essential extension of its socle, then it is $\mathcal{D}$-torsion.

(v) For any left R-module M, its torsion part M_t is the unique maximal essential extension in M of its socle $soc(M)$.

We will be looking for cases in which the Dickson torsion theory is stable for a left Noetherian ring R. There are two such cases which imply the stability of the Dickson torsion theory, but first we should introduce a partial order among the torsion theories defined on R-mod.

For a ring R we denote the family of all hereditary torsion theories defined on R-mod by R-tors. Note that R-tors corresponds bijectively to a set, because each hereditary torsion theory can be identified with its Gabriel filter, which is an element of the power set of all left ideals of R, see for example [20, Proposition 4.6]. We define a partial order in R-tors with the help of the following result:

Proposition 5.2.5 *[20, Proposition 2.1] For torsion theories τ and σ on R-mod the following conditions are equivalent:*

(a) Every τ-torsion left R-module is σ-torsion;

(b) Every σ-torsionfree left R-module is τ-torsionfree.

In case τ and σ are torsion theories on R-mod which satisfy the equivalent conditions of the above proposition, we say that τ is a **specialization** of σ and that σ is a **generalization** of τ. We denote this situation by $\tau \leq \sigma$. This defines a partial order in R-tors. For example, with respect to this ordering, the Goldie torsion theory is the smallest torsion theory in which every cyclic singular left R-module is torsion and the Dickson torsion theory is the smallest torsion theory in which every simple left R-module is torsion.

5.3 Cyclic singular modules with nonzero socle

We now give a sufficient condition for a torsion theory to be stable. The following result is implicitly stated in [65]. We record it as a lemma and give its proof for completeness.

Lemma 5.3.1 *Any generalization of Goldie's torsion theory $\mathcal{G}$ is stable.*

Proof: Suppose that (T,F) is a torsion theory which is a generalization of $\mathcal{G}$. For all $M \in T$, since M is essential in its injective hull $E(M)$, $E(M)/M$ is Goldie torsion. Since $T_{\mathcal{G}} \subseteq T$, $E(M)/M$ also belongs to T. Since T is closed under extensions, it follows that $E(M)$ also belongs to T and hence (T,F) is stable. □

In particular, the Dickson torsion theory is stable if it is a generalization of Goldie's torsion theory. This can be summarized as follows:

Corollary 5.3.2 *If every cyclic singular left R-module has a nonzero socle then the Dickson torsion theory is stable.*

Proof: By assumption, every cyclic singular left R-module has the property that every epimorphic image has nonzero socle. Thus every such module belongs to the Dickson torsion class. Since the Goldie torsion theory is the smallest torsion theory in which every cyclic singular module is torsion, it follows that the Dickson torsion theory is a generalization of Goldie torsion theory, hence it is stable. □

The rings R such that every cyclic singular left R-module has a nonzero socle are called **C-rings** in [55]. We will now give a list of different characterizations of C-rings, but we first give the necessary definitions. A submodule $K \leq L$ of a module L is called **neat** if any simple module S is projective relative to the projection $L \to L/K$. Also, a submodule A of a module B is called **closed** if A has no proper essential extensions in B. Note that a closed submodule is always neat. Lastly, if $\mathcal{C}$ is a nonempty collection of left ideals of a ring R, we say that a left R-module M is $\mathcal{C}$-**injective** if for every $I \in \mathcal{C}$, every R-homomorphism $I \to M$ can be lifted to an R-homomorphism $R \to M$.

Proposition 5.3.3 *[12, 10.10][62, Lemma 4] Let R be a ring and $\mathcal{M}ax$ be the collection of maximal left ideals of R. The following conditions are equivalent.*

(a) Every neat left ideal of R is closed.

(b) A left ideal of R is closed if and only if it is neat.

(c) For every left R-module, closed submodules are neat.

(d) R is a C-ring.

(e) Every singular module is semi-Artinian.

(f) Every $\mathcal{M}ax$-injective left R-module is injective.

Example 5.3.4 *Hereditary Noetherian rings are C-rings [47, 5.4.5] and hence they have property $(\diamond)$. While it is true that every Noetherian C-ring has property $(\diamond)$, there are*

Noetherian rings which have property $(\diamond)$ but are not C-rings. For example, the ring of polynomials $R = K[x, y]$ in two indeterminates over a field K is a commutative Noetherian domain and hence has property $(\diamond)$. The ideal $I = \langle x \rangle$ is essential in R since R is a domain, but the singular module $M = R/I$ has zero socle and hence R is not a C-ring.

Another class of rings over which the cyclic singular modules have nonzero socle is the so called class of SI-rings. A ring R is said to be a **left SI-ring** if every singular left R-module is injective. SI-rings satisfy the stronger property that R/E is semisimple for every essential left ideal E of R [19, 17.2]. Moreover, each left SI-ring is left hereditary [19, 17.1]. The converse is true if the ring is additionally a left SC-ring, that is, if every singular module is continuous. Recall that a module M is π**-injective** if it is fully invariant in its injective hull $E(M)$ for every endomorphism of $E(M)$. The module M is called **direct injective** if for every direct summand X of M, the monomorphism $X \to M$ splits. Finally, the module M is called **continuous**, if it is π-injective and direct injective (see [19, §1.2.]). See [19, 17.4] also for a list of equivalent conditions for a ring R to be a left SI-ring.

5.4 Soclefree modules containing nonzero projective submodules

As a second sufficient condition for the stability of the Dickson torsion theory, we consider the following property of an arbitrary torsion theory τ:

(P) Every τ-torsionfree left R-module contains a nonzero projective submodule.

Teply considered the property (P) in [65] and he proved the following result.

Lemma 5.4.1 *[65, Proposition 1] Any torsion theory τ which has the property (P) is a generalization of $\mathcal{G}$.*

In other words, the Goldie torsion theory is the smallest torsion theory which possibly satisfies the condition (P). In particular, the property (P) is sufficient for a torsion theory τ to be a generalization of the Goldie torsion theory, therefore making it stable by Lemma 5.3.1.

Applying this new property to the Dickson torsion theory, we get:

Corollary 5.4.2 *The Dickson torsion theory is stable if every soclefree left R-module contains a nonzero projective submodule.*

If R is a simple Noetherian C-ring, then every hereditary torsion theory in R-Mod has the (P) property [65, Theorem 1(3)(b)]. On the other hand, if R is a ring such that the Dickson torsion theory has the property (P), then R is a C-ring. Because, as we noted in Lemma 5.4.1, in this case the Dickson torsion theory is the generalization of the Goldie torsion theory and so every singular module is semi-Artinian, *i.e.* R is a C-ring.

Let U be a nonempty subset of R-tors. We define the join $\vee U$ of the set U by declaring a left R-module to be $\vee U$-torsionfree if and only if it is τ-torsionfree for all $\tau \in U$ [20, Proposition 2.6]. For a simple left R-module S, let τ_S be the torsion theory generated by S which is the smallest torsion theory in which S is torsion. Then the Dickson torsion theory is the join $\vee \tau_S$ where the join is taken over all nonisomorphic simple left R-modules. Hence for each simple left R-module S we have $\tau_S \leq \mathfrak{D}$, and it follows that if τ_S satisfies (P) for a simple left R-module S then the Dickson torsion theory is stable.

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